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Research Article

On Extremal Problems in Certain New Bergman Type Spaces in Some Bounded Domains in $\mathbb{C}^n$

Romi F. Shamoyan[1] and Olivera Mihić[2]

[1] *Department of Mathematics, Bryansk State Technical University, Bryansk 241050, Russia*
[2] *Fakultet organizacionih nauka, Jove Ilića 154, 11000 Belgrade, Serbia*

Correspondence should be addressed to Olivera Mihić; oliveradj@fon.rs

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Based on recent results on boundedness of Bergman projection with positive Bergman kernel in analytic spaces in various types of domains in $\mathbb{C}^n$, we extend our previous sharp results on distances obtained for analytic Bergman type spaces in unit disk to some new Bergman type spaces in Lie ball, bounded symmetric domains of tube type, Siegel domains, and minimal bounded homogeneous domains.

1. Introduction

The goal of this paper is to add several new results for distances in analytic Bergman type spaces of functions of several variables. It turns out that our distance theorem we proved before in case of unit disk, a sharp result under certain natural additional condition, is valid also in various domains and various Bergman type analytic spaces. Namely, we look at analytic Bergman type spaces in Lie ball, bounded symmetric domains of tube type, Siegel domains of second type, and minimal bounded homogeneous domains. These analytic spaces act as direct extensions of well-known analytic Bergman spaces in the unit disk. These analytic spaces are relatively new and we will include some basic facts on them in our paper. They will also be needed for proofs of our assertions partially. We will start this paper with two sharp results on distances in Bergman type spaces in two domains: Siegel domains of second type and in minimal homogeneous domains in $\mathbb{C}^n$. Then one side estimates for distance function in Lie ball and bounded symmetric domains of tube type will be given based directly on recent advances related to boundedness of Bergman type projections in Bergman type spaces in these type domains.

Our intention in this paper is the same as in our previous papers on this topic. Namely, we collect some facts from earlier investigation concerning Bergman projection with positive Bergman kernel and Bergman kernel and use them for our purposes in estimates of $\operatorname{dist}_Y(f, \mathscr{X})$ function (distance function).

Following our previous papers [1, 2] we can easily obtain a sharp result for distance function. We need only several tools and the following scheme.

First we need an embedding of our quazinormed analytic space (in any domain) into another one ($X \subset Y$); this immediately poses a problem of $\operatorname{dist}_Y(f, X) = \inf_{g\in X}\|f - g\|_Y$ for all $f \in Y \setminus X$. Then we need the Bergman reproducing formula for all f function from Y space. Then, finally, we use the boundedness of Bergman type projections with $|K(z,w)|$ positive kernel acting from X to X together with Forelli-Rudin type sharp estimates of Bergman kernel. These three tools were used in general Siegel domain of second type, polydisk, and unit ball in [1–4] (see also various references there). We continue to use these tools providing new sharp (and not sharp) results in various spaces of analytic functions in this paper.

Note that our theorem on Siegel domains was formulated in [5] without proof. We provide the complete proof here. We also note that various problems, related to Bergman type projections, are applied in many problems in function theory (see, e.g., [6] and references there).

First we provide a known result in the unit disk with complete proof taken from our previous papers [1, 2]. In the unit disk case all arguments here are short and transparent and are based on several tools like Forelli-Rudin type estimate and estimates for Bergman type projections with positive Bergman kernel. Then we will see arguing similarly as in unit disk and we will easily complete the proof of more complicated cases. The complete formulations of our last theorems will be given, but sometimes sketches of proofs will be added and details of proofs of higher-dimensional cases will be left to readers.

Note that it is easy to see that our assertions may have various applications in approximation theory; for example see [6] and references there.

The base of our proofs is properties of Bergman projection in various domains given in [7–12]. The estimates of Bergman kernel from [7–12] are also playing an important role below in our proofs. Note that arguments we use below are very close to arguments which were used before in [1, 2, 13]. As a result we alert the reader that the exposition is sketchy sometimes.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation $A \asymp B$ means that there is a positive constant C, such that $B/C \leq A \leq CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

This paper can be viewed as continuation of [13] where various other cases (domains) were also considered. In [13] the base of all our proofs in complex domains in higher dimension was the Bergman reproducing formula, while here all our assertions are based on some recent results on boundedness of Bergman projections with positive Bergman kernel in Bergman spaces in such type domains.

2. Notations, Definitions, and Preliminaries

We will need various definitions and assertions for formulations of main results. These are assertions on various types of domains we consider in this paper and analytic functions on them.

We denote by $\mathbf{B}_n$ the unit ball in $\mathbb{C}^n$. As usual, we denote by $H(\mathbf{B}_n)$ the space of all holomorphic functions in $\mathbf{B}_n$. For $1 \leq p < +\infty$ and $\alpha > -1$, denote by $H^p_\alpha(\mathbf{B}_n)$ (or $A^p_\alpha(\mathbf{B}_n)$) the space of all functions f holomorphic in $\mathbf{B}_n$ and satisfying the condition

$$\int_{\mathbf{B}_n} |f(w)|^p \left(1 - |w|^2\right)^\alpha d\nu(w) < +\infty, \tag{1}$$

where $d\nu$ is the Lebesgue measure in $\mathbb{C}^n$.

Further, for a complex number β with $\operatorname{Re}\beta > -1$, put

$$c_n(\beta) = \frac{\Gamma(n+1+\beta)}{\pi^n \cdot \Gamma(1+\beta)}. \tag{2}$$

Let $A^\infty_\alpha(\mathbf{B}_n) = \{f \in H(\mathbf{B}_n) : \sup |f(z)|(1-|z|)^\alpha < \infty\}$, $\alpha \geq 0$.

The following theorem is well-known and it has many applications in various problems in function theory (see [14]).

Theorem A. *Assume that $1 \leq p < +\infty$, $\alpha > -1$ and that the complex number β satisfies the condition*

$$\begin{gathered} \operatorname{Re}\beta \geq \alpha, \quad p = 1 \\ \operatorname{Re}\beta > \frac{\alpha+1}{p} - 1, \quad 1 < p < \infty. \end{gathered} \tag{3}$$

Then each function $f \in A^p_\alpha(\mathbf{B}_n)$ admits the following integral representations:

$$f(z) = c_n(\beta) \cdot \int_{\mathbf{B}_n} \frac{f(w)\left(1-|w|^2\right)^\beta}{\left(1 - \langle z, w\rangle\right)^{n+1+\beta}} d\nu(w), \quad z \in \mathbf{B}_n, \tag{4}$$

$$\overline{f(0)} = c_n(\beta) \cdot \int_{\mathbf{B}_n} \frac{\overline{f(w)}\left(1-|w|^2\right)^\beta}{\left(1 - \langle z, w\rangle\right)^{n+1+\beta}} d\nu(w), \quad z \in \mathbf{B}_n, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian inner product in $\mathbb{C}^n$.

For $n > 1$, the theorem was proved in [15] (when $\alpha = 0$) and in [16] (when $\alpha > -1, n = 1$).

These integral representation theorems were the core of our approach for estimates of distance function in our recent papers (see [1, 2, 13, 17]) and we will see the same in this paper.

We will start this section with various known assertions on analytic function spaces on Siegel domains of second type. Next we will continue adding some lemmas on each Bergman type analytic space on each domain in higher dimension which we will consider in this paper. We alert the reader that some assertions below will serve as introductory material and will not be used during the proof to make the reading of paper more convenient.

Let D be a usual homogeneous Siegel domain of second type. Let $d\nu$ denote the Lebesgue measure on D (for all other bounded domains in this paper we will also use the same notation below) and let as usual $H(D)$ be the space of holomorphic functions on D endowed as usual with the topology of uniform convergence on compact subsets of D.

The Bergman projection P of D is as usual the orthogonal projection of $L^2(D, d\nu)$ onto its subspace $A^2(D)$ consisting of holomorphic functions. Moreover it is known that P is the integral operator defined on $L^2(D, d\nu)$ by the Bergman kernel $B(z, \zeta)$ which for D was computed for example in [18, 19].

Let r be a real number, for example. We fix it. Since D is homogeneous, the function $\zeta \to B(\zeta, \zeta)$ does not vanish on D; we can set

$$L^{p,r}(D) = L^p\left(D, B^{-r}(\zeta, \zeta)\, d\nu(\zeta)\right), \quad 0 < p < \infty. \tag{6}$$

Let p be an arbitrary positive number. The weighted Bergman space is defined as usual by $A^{p,r}(D) = L^{p,r}(D) \cap H(D)$. We put $A^{p,0} = A^p(D)$.

The so-called weighted Bergman projection P_ε is the orthogonal projection of $L^{2,\varepsilon}(D)$ onto $A^{2,\varepsilon}(D)$. This fact can be found in [8, 10]. It is proved in [8, 10] that there exists a real number $\varepsilon_D < 0$ such that $A^{2,\varepsilon}(D) = \{0\}$ if $\varepsilon \leq \varepsilon_D$ and that for $\varepsilon > \varepsilon_D$. P_ε is the integral operator defined on $L^{2,\varepsilon}(D)$ by the weighted Bergman kernel $C_\varepsilon B^{1+\varepsilon}(\zeta, z)$. In all our work we will assume that $\varepsilon > \varepsilon_D$.

The norm $\|\cdot\|_{p,r}$ of $A^{p,r}(D)$ with $r > \varepsilon_D$ is defined by

$$\|f\|_{p,r} = \left(\int_D |f(z)|^p B^{-r}(z,z)d\nu(z)\right)^{1/p}, \quad f \in A^{p,r}(D). \tag{7}$$

We need some assertions (see [8, 10]).

Note the exact expression of Bergman kernel for these domains can be seen in [8, 10, 18].

We denote by $b((\tau_1,\tau_2),(\tau_3,\tau_4))$ Bergman kernel for the Siegel domain of the second type, which differs from $B(z,\zeta)$ Bergman kernel by constant. We will use it in text also.

Lemma A. *Let $h \in L^\infty(D)$. Take $\rho > \rho_0$ for large fixed ρ_0. Then the function*

$$z \longrightarrow G(z) = \int_D B^{1+\rho}(z,\zeta)\,h(\zeta)\,d\nu(\zeta) \tag{8}$$

satisfies the estimate $\sup_{z\in D}|G(z)|B^{-\rho}(z,z) \le C\|h\|_\infty$ and $G \in H(D)$.

Lemma B. *For each ρ sufficiently large and for each $G \in H(D)$ such that $\sup_{z\in D}|G(z)||B^{-\rho}(z,z)| < \infty$ one has the reproducing formula*

$$G(\zeta) = C_\rho \int_D B^{1+\rho}(\zeta,z)\,G(z)\,B^{-\rho}(z,z)\,d\nu(z), \quad z \in D. \tag{9}$$

We will need for our theorems some basic facts for Siegel domains of second type. We denote by d_i, q_i, and n_i parameters of a Siegel domains of second type (see [4, 5, 8, 10]). We will use usual operations between two vectors for such parameters below in our text.

The following lemma is complete analogue of so-called Forelly-Rudin type estimates for our Siegel domains of second type (see [8, 10]).

Lemma C. *Let α and ε be in $\mathbb{R}^l$, $(\zeta,\nu) \in D$. Then for $\varepsilon_i > (n_i+2)/2(2d-q)_i$ and $\alpha_i - \varepsilon_i > n_i/(-2)(2d-q)_i$, $i = 1,\dots,l$*

$$\begin{aligned}\int_D \left|B^{1+\alpha}((\zeta,\nu),(z,u))\right| B^{-\varepsilon}((z,u),(z,u))\,d\nu(z,u) \\ = C_{\alpha,\varepsilon}B^{\alpha-\varepsilon}((\zeta,\nu),(\zeta,\nu)).\end{aligned} \tag{10}$$

Lemma D. *Let r be a vector of $\mathbb{R}^l$ such that $r_i > (n_i+2)/(-2)(2d-q)_i$ for all $i = 1,\dots,l$ and p is a real number such that $1 \le p < \min\{(n_i - 2(2d-q)_i(1+r_i))/n_i\}$. Then for all $\varepsilon \in \mathbb{R}^l$ such that $\varepsilon_i > (n_i+2)/(2(2d-q)_i)((p-1)/p) + (r_i/p)$, $i = 1,\dots,l : P_\varepsilon f = f$, $f \in A^{p,q}$.*

We list in Lemma E other properties of Bergman kernel. The last estimate in assertion below is an embedding theorem which connect so-called growth spaces with Bergman spaces. This allows to pose a distance problem (see also the complete analogue of this result in other simpler domains in [1, 17]).

Lemma E. *Let $\alpha \in \mathbb{R}^l$, $\alpha_j \le 0$, $i = 1,\dots,l$. Then $|b^\alpha((\zeta,\nu),(z,u))| \le c_\alpha b^\alpha((\zeta,\nu),(\zeta,\nu))$ and $|b^\alpha((\zeta,\nu)+(\zeta',\nu'),(z,u)+(z',u'))| \le c_\alpha b^\alpha((\zeta,\nu),(\zeta,\nu))$ for all (ζ,ν), (ζ',ν'), (z,u), (z',u') in D. For all $f \in A^{p,r}(D)$, $p > 0$*

$$|f(z,u)|^p \le Cb^{1+r}((z,u),(z,u))\,\|f\|^p_{p,r}. \tag{11}$$

The following result concerns the boundedness of Bergman type projection with positive Bergman kernel in weighted Bergman spaces. Note that this fact is classical in simpler domains and it has also many applications in analytic function theory.

Proposition A. *Let ε and r be in $\mathbb{R}^l$ such that $\varepsilon_i > 1/(2d-q)_i$ and $r_i > (n_i+2)/2(2d-q)_i$, $i = 1,\dots,l$. Then P_ε is bounded from $L^{p,r}(D)$ into $A^{l,r}(D)$ if*

$$\begin{aligned}\max_{i=1,\dots,l}\left\{1, \frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i + 2 - 2(2d-q)_i\varepsilon_i}\right\} \\ < p < \min_{i=1,\dots,l}\frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i}.\end{aligned} \tag{12}$$

The following assertion provides integral representation for a certain so-called analytic "growth space" on Siegel domains of the second type.

Proposition B. *Let r and ε be two vectors of $\mathbb{R}^l$ such that $\varepsilon_i > n_i/(-2)(2d-q)_i$, $r_i > ((n_i+2)/2(2d-q)_i) + \varepsilon_i$, $i = 1,\dots,l$. Let G be in $H(D)$ such that*

$$\sup_{z\in D}\{|G(z)|\,B^{-\varepsilon}(z,z)\} < \infty; \tag{13}$$

then $PrG = G$.

The following result explains the structure of functions from Bergman spaces on Siegel domains of second type. It is an extension of a classical theorem on atomic decomposition of Bergman spaces in the unit disk on a complex plane.

Proposition C. *Let $D \subset \mathbb{C}^N$ be a symmetric Siegel domain of second type, $p \in (2N/(2N+1),1)$, $z \in \mathbb{R}^l$, $r_j > (n_i+2)/2(2d-q)_i$. Then there are two constants $C = C(p,r)$ and $C_1 = C_1(p,r)$ such that for every $f \in A^{p,r}(D)$ there exists an l^p sequence $\{\lambda_i\}$ such that*

$$f(z) = \sum_{i=0}^{\infty}\lambda_i b^{\alpha/p}(z,z_i)\,b^{(1+r-\alpha)/p}(z_j,z_j), \tag{14}$$

where $\{z_i\}$ is a lattice in D and the following estimate holds:

$$C\|f\|^p_{p,r} \le \varepsilon|\lambda_i|^p \le C\|f\|^p_{p,r}. \tag{15}$$

We add some basic facts on minimal bounded homogeneous domains and we will use them partially in our paper (see [11, 12]).

Let D be a bounded domain in $\mathbb{C}^n$. We say that D is a minimal domain with a center $t \in D$ if the following condition is satisfied: for every biholomorphism $\psi : D \to D'$ with $\det J(\psi,t) = 1$, (Js is the complex Jacobian of the map ψ), we have

$$\mathrm{Vol}(D') \ge \mathrm{Vol}(D). \tag{16}$$

Let now us denote by K_D the Bergman kernel of D, that is, the reproducing kernel of $L^2(D)$. It is known that D is a minimal domain with a center t if and only if $K_D(z,t) = K_D(t,t)$ for any $z \in D$, (see [20] and references there).

From [21], Proposition 3.6, we see that D is a minimal domain with a center t if and only if

$$K_D(z,t) = \frac{1}{\operatorname{Vol}(D)}, \tag{17}$$

for any $z \in D$.

Every bounded homogeneous domain is biholomorphic to a representative bounded homogeneous domain.

Therefore, every bounded homogeneous domain is biholomorphic to a minimal bounded homogeneous domain (see [11, 12]).

For any $z \in \mathcal{U}$ and $\rho > 0$, let

$$\mathbf{B}(z,\rho) = \{w \in \mathcal{U} : \beta(z,w) \le \rho\} \tag{18}$$

be the Bergman metric disk with center z and radius ρ, where β denotes the Bergman distance on $\mathcal{U}$, (see [11, 12]).

We fix a minimal bounded homogeneous domain $\mathcal{U}$ with a center t. For a bounded linear operator T on $A^2_\alpha(\mathcal{U})$ (this is standard analytic part of standard L^2 space in $\mathcal{U}$) the Berezin symbol $\widetilde{T}$ of T is defined by

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle \ (z \in \mathcal{U}), \tag{19}$$

where k_z is a normalized Bergman kernel in Bergman space A^2_α in minimal bounded homogeneous domain $\mathcal{U}$. For a Borel measure μ on $\mathcal{U}$, we define a function $\widetilde{\mu}$ on $\mathcal{U}$ by

$$\widetilde{\mu}(z) = \int_{\mathcal{U}} |k_z(w)|^2 d\mu(w), \tag{20}$$

which is called the Berezin symbol of the measure μ. Since $K_{\mathcal{U}}(z,w)$ is a bounded function on $\mathbf{B}(t,\rho) \times \mathcal{U}$, $\widetilde{\mu}$ is a continuous function if μ is finite.

We will provide some basic facts for a minimal bounded homogeneous domains.

Lemma F (see [20]). *There exists a constant M_ρ such that*

$$M_\rho^{-1} \le |k_a(z)|^2 \mathit{Vol}(\mathbf{B}(a,\rho)) \le M_\rho, \tag{21}$$

for all $a \in \mathcal{U}$ and $z \in \mathbf{B}(a,\rho)$.

Lemma G (see [20]). *There exists a sequence $\{w_j\} \subset \mathcal{U}$ satisfying the following conditions.*

(1) $\mathcal{U} = \bigcup_{j=1}^{\infty} \mathbf{B}(w_j,\rho)$.

(2) $\mathbf{B}(w_i,\rho/4) \cap \mathbf{B}(w_j,\rho/4) = 0$.

(3) *There exists a positive integer N such that each point $z \in \mathcal{U}$ belongs to at most N of the sets $\mathbf{B}(w_j,2\rho)$.*

Lemma H (see [20]). *There exists a constant C such that*

$$|f(a)|^p \le \frac{C}{\mathit{Vol}(\mathbf{B}(a,\rho))} \int_{\mathbf{B}(a,\rho)} |f(z)|^p d\nu(z) \tag{22}$$

for all $f \in H(\mathcal{U})$, $p \ge 1$, and $a \in \mathcal{U}$, where $H(\mathcal{U})$ is a space of analytic functions in $\mathcal{U}$.

Theorem B (see [22]). *Take any $\rho > 0$. Then, there exists $C_\rho > 0$ such that*

$$C_\rho^{-1} \le \left|\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)}\right| \le C_\rho, \tag{23}$$

for all $z, a \in \mathcal{U}$ with $\beta_{\mathcal{U}}(z,a) \le \rho$, where $\beta_{\mathcal{U}}$ means the Bergman distance on $\mathcal{U}$.

Authors in [22] introduce certain equivariant holomorphic maps $\theta_{n_j} : \mathcal{U} \to \mathcal{U}'$ for $j = 1,\dots,r$ (:=rank $\mathcal{U}$) from $\mathcal{U}$ into the Siegel disk $\mathcal{U}_n$ of rank n_j. Authors in [22] obtain the following formula for the description of $K_{\mathcal{U}}$.

Theorem C (see [22]). *There exist integers $s_1,\dots,s_r$ such that*

$$K_{\mathcal{U}}(z,w) = \mathit{Vol}(\mathcal{U})^{-1} \prod_{j=1}^{r} \left\{\det\left(I_{n_j} - \theta_{n_j}(z)\overline{\theta_{n_j}(w)}\right)\right\}^{-s_j} \tag{24}$$

for $z, w \in \mathcal{U}$. Recall that the Bergman kernel $K_{\mathcal{U}_m}$ of the Siegel disk $\mathcal{U}_m$ is given by

$$K_{\mathcal{U}_m}(z,w) = \mathit{Vol}(\mathcal{U}_m)^{-1} \det\left(I_m - z\overline{z}\right)^{-(m+1)}. \tag{25}$$

We will denote the weighted reproducing Bergman kernel for weighted Bergman A^2_α spaces in this type domains below simply as K_α omitting index $\mathcal{U}$.

Note from lemmas above (see [11]) that we have

$$\sup |f(z)| K(z,z)^t < C|f|_{A^2_\beta}, \quad \text{where } t = \frac{(-\beta-1)}{2}. \tag{26}$$

This allows putting distance problems for these domains which we solve in Theorem 8.

We need now some preliminaries for Bergman spaces in Lie ball.

Let D denote each of the following domains in $\mathbb{C}^n$, $n \ge 3$:

(1) the tube $\Omega = \mathbb{R}^n + i\Gamma$ over the spherical cone

$$\Gamma = \left\{(y_1,\dots,y_n) \in \mathbb{R}^n : y_1 > 0, y_1 y_2 - y_3^2 - \cdots - y_n^2 > 0\right\}, \tag{27}$$

(2) the Lie ball

$$\omega = \left\{z \in \mathbb{C}^n : \left|\sum_{j=1}^{n} z_j^2\right| < 1, 1 - 2|z|^2 + \left|\sum_{j=1}^{n} z_j^2\right|^2 > 0\right\}. \tag{28}$$

Obviously, the first domain is unbounded while the second one is bounded. It is well- known that they are biholomorphically equivalent and, in Elie Cartans classification of bounded symmetric domains [23], they are representatives of class IV (according to Huas numbering [24]). We are interested in bounded Lie ball, bounded symmetric domains of tube type, Siegel domains of second type, and minimal bounded homogeneous domains in this paper.

To make the exposition easier, we remind the readers about the basic definitions again.

Let $H(D)$ denote the space of holomorphic functions in D domain and let, as above, $d\nu$ be Lebesgue measure in $\mathbb{C}^n$. For every $p \geq 1$, the Bergman space $A^p(D)$ is defined by $A^p(D) = H(D) \cap L^p(D, d\nu)$. For every $f \in A^p(D)$, we set $\|f\|_{A^p(D)} = \|f\|_{L^p(D,d\nu)}$ for $p \geq 1$; this is a norm under which $A^p(D)$ is a Banach space. The Bergman projection P_D of D is the orthogonal projection of the Hilbert space $L^2(D, d\nu)$ onto its closed subspace $A^2(D)$. Moreover, P_D is the integral operator associated with a kernel $B_D(\cdot,\cdot)$ called the Bergman kernel of D. Finally, let P_D^* denote the integral operator associated with the positive kernel $|B_D(\cdot,\cdot)|$.

The following results were proved in [7].

Theorem D (see [7]). *For every $p \in (1, (3n-2)/2n] \cup [(3n-2)/(n-2), \infty)$, the Bergman projection P_D is unbounded on $L^p(D, d\nu)$.*

Theorem E (see [7]). *Let $p \geq 1$. The operator P_D^* is bounded on $L^p(D, d\nu)$ if and only if $p \in ((2n-2)/n, (2n-2)/(n-2))$. Furthermore, the Bergman projection P_D is bounded from $L^p(D, d\nu)$ to $A^p(D)$ when $p \in ((2n-2)/n, (2n-2)/(n-2))$.*

For the tube domain Ω some of these results were announced in [25]. The question whether P_D is bounded on $L^p(D, d\nu)$ when p belongs to $((3n-2)/2n, (2n-2)/n] \cup [(2n-2)/(n-2), (3n-2)/(n-2))$ remains open. The case of all homogeneous Siegel domains of second type has recently been considered by D. Bekolle and A. Temgoua Kagou. They proved that there is a range of p, around 2, where the Bergman projection is bounded in L^p, while there is a range of p, around 1 and ∞, where it is unbounded (see [8]). In all cases the critical result is not known.

We add some basic facts on Bergman kernel on these domains.

Proposition D (see [7]). *The Bergman kernel $B_\Omega(s, z)$ of Ω is given by*

$$B_\Omega(\zeta, z) = c_n \left[(\zeta_1 - \overline{z}_1)(\zeta_2 - \overline{z}_2) - \sum_{j=3}^{n} (\zeta_j - \overline{z}_j)^2 \right]^{-n}, \quad (29)$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)$, $z = (z_1, \ldots, z_n) \in \Omega$.

Definition 1 (see [7]). Let $k(t, y)$ denote the positive kernel defined on the cone $\Gamma k(t, y) = [(t_1 + y_1)(t_2 + y_2) - \sum_{j=3}^{n}(t_j + y_j)^2]^{-n/2}$, $t = (t_1, \ldots, t_n)$, $y = (y_1, \ldots, y_n)$, $y \in \Gamma$.

Proposition E (see [7]). *For each $p \geq 1$, there exists a constant C_p such that, for all $y \in \Gamma$ and $\xi = s + it \in \Omega$,*

$$\int_{\mathbb{R}^n} |B_\Omega(\zeta, x + iy)|^p dx = C_p [k(t, y)]^{2p-1}. \quad (30)$$

Moreover, there exists a constant c_p such that, for each $y \in \Gamma$ such that $|y| < 1/100$ and each $\zeta = s + it \in \Omega$ such that $|\zeta| < 1/100$,

$$\int_{I \times \cdots \times I} |B_\Omega(\zeta, x + iy)|^p dx \geq C_p [k(t, y)]^{2p-1}, \quad (31)$$

where I denotes the interval $[-1, 1]$.

Let $z = \Phi(z')$ be the linear fractional mapping from ω onto Ω which is given in [23]. In particular, we assume that $\Phi(0) = e$, where $e = (i, i, 0, \ldots, 0)$ and Φ is holomorphic outside $Z = \{z \in \mathbb{C}^n : Q(z) = 0\}$, where Q is a polynomial such that $Q(0) = 1$. In view of the change of variables formula, the Bergman kernel $B_\omega(\zeta', z')$ of ω has the following expression in terms of that of Ω:

$$B_\omega(\zeta', z') = B_\Omega(\Phi(\zeta'), \Phi(z')) J\Phi(\zeta') \overline{J\Phi(z')}, \quad (32)$$

where Js is the complex Jacobian of the map Φ. On the other hand, since ω is a circular domain, for each real number θ,

$$B_\omega(e^{i\theta}\zeta', e^{i\theta}z') = B_\omega(\zeta', z') \quad (33)$$

and thus, there exists a constant C such that $B_\omega(\zeta', 0) = C$ for each $\zeta' \in \omega$. Hence, from (32), we get

$$J\Phi(\zeta') = C' [B_\omega(\Phi(\zeta'), e]^{-1}. \quad (34)$$

The following property of Bergman kernel is vital.

Lemma I (see [7]). *For all z and ζ in Ω,*

$$|B_\omega(\zeta, z)| \leq B_\omega(z, z). \quad (35)$$

We add some basic facts on bounded symmetric domains of tube type from [23].

Let $\widetilde{D}$ be an irreducible bounded symmetric domain of tube type in $\mathbb{C}^n$. That is, $\widetilde{D}$ is conformally equivalent to a tube domain $\widetilde{T}_{\widetilde{\Omega}} = \mathbb{R}^n + i\widetilde{\Omega}$ over a symmetric cone $\widetilde{\Omega}$ in $\mathbb{R}^n$. Irreducible symmetric cones are completely classified (see [26]), being either light-cones

$$\Lambda_n = \{(y_1, y') \in \mathbb{R}^n : y_1 > |y'|\}, \quad n \geq 3, \quad (36)$$

or cones of positive-definite symmetric or hermitian matrices, namely,

$$\mathrm{Sym}_+(r, \mathbb{R}), \qquad \mathrm{Her}_+(r, \mathbb{C}), \qquad \mathrm{Her}_+(r, \mathbb{H}), \qquad \mathrm{Her}_+(3, \mathbb{O}). \quad (37)$$

We write r for the rank of the cone (which in light-cones is $r = 2$) and Δ for the associated determinant function (which in light-cones is the Lorentz form $\Delta(y) = y_1^2 - |y'|^2$).

An important open question in these domains, $\widetilde{D}$ and $\widetilde{T}_{\widetilde{\Omega}}$, concerns the L^p boundedness of the associated Bergman projections, that is, the orthogonal projection P mapping L^2 into the subspace of holomorphic functions A^2. In contrast with Cauchy-Szego projections (which are not bounded in L^p for any $p \neq 2$, if $n > 1$), the L^p-boundedness of Bergman projections has been conjectured in a small interval around $p = 2$, namely,

$$1 + \frac{n-r}{2n} < p < 1 + \frac{2n}{n-r}. \quad (38)$$

At the moment, positive results are only known to hold in a proper subinterval

$$1 + \frac{n-r}{2n-r} < p < 1 + \frac{2n-r}{n-r}, \quad (39)$$

with a small improvement over this range in the case of light-cones.

Also, we are interested in applying the transference principle to the family of weighted Bergman projections in $\widetilde{D}$ and $\widetilde{T}_{\widetilde{\Omega}}$. Using the notation in the text from [26], Chapter XIII, these operators are defined for $\nu > (2n/r) - 1$ by

$$\begin{gathered} P_\nu f(z) = \int_{\widetilde{D}} B_\nu^{\widetilde{D}}(z,w) f(w)\, d\mu_\nu(w), \quad z \in \widetilde{D}, \\ \mathscr{P}_\nu g(\zeta) = \int_{\widetilde{T}_{\widetilde{\Omega}}} B_\nu^{\widetilde{T}_{\widetilde{\Omega}}}(\zeta,\eta)\, g(\eta)\, d\lambda_\nu(\eta), \quad \zeta \in \widetilde{T}_{\widetilde{\Omega}}, \end{gathered} \tag{40}$$

where the Bergman kernels and their associated measures have the explicit expressions

$$\begin{gathered} B_\nu^{\widetilde{D}}(z,w) = c_\nu h(z,w)^{-\nu}, \qquad d\mu_\nu(w) = h(w)^{\nu-(2n/r)} dw, \\ B_\nu^{\widetilde{T}_{\widetilde{\Omega}}}(\zeta,\eta) = c_\nu' \Delta(\zeta - \overline{\eta})^{-\nu}, \qquad d\lambda_\nu(\eta) = \Delta(\Im m\eta)^{\nu-(2n/r)} d\eta \end{gathered} \tag{41}$$

for certain constants c_ν, c_ν'. Here $h(z,w)$ denotes the unique polynomial (holomorphic in z and antiholomorphic in w) such that $h(z) := h(z,z)$ is U-invariant and $h(x) = \Delta(e - x^2)$, $x \in \mathbb{R}^n$.

The Bergman kernels in these two domains are related by the formula

$$B_\nu^{\widetilde{D}}(z,w) = \widetilde{c}_\nu B_\nu^{\widetilde{T}_{\widetilde{\Omega}}}(\Phi(z), \Phi(w)) \left(J_\Phi(z)\overline{J_\Phi(w)}\right)^{\nu/(2n/r)}. \tag{42}$$

Corollary 2 (see [9]). *Let $\nu > (2n/r) - 1$ and $p, q \in [1,\infty]$. Then the following are equivalent:*

(a) *P_ν is bounded from $L^p(\widetilde{D}, d\mu_\nu) \to L^q(\widetilde{D}, d\mu_\nu)$;*

(b) *$\mathscr{P}_\nu^{\mathbf{B}}$ is bounded from $L^p(\widetilde{T}_{\widetilde{\Omega}}, d\lambda_\nu) \to L^q(\widetilde{T}_{\widetilde{\Omega}}, d\lambda_\nu)$.*

Remark 3. The statements in the corollary can only hold if $p \geq q$.

The transference principle also applies to the positive operators

$$\begin{gathered} P_\nu^+ f(z) = \int_{\widetilde{D}} \left| B_\nu^{\widetilde{D}}(z,w) \right| f(w)\, d\mu_\nu(w), \\ \mathscr{P}_\nu^+ g(\zeta) = \int_{\widetilde{T}_{\widetilde{\Omega}}} \left| B_\nu^{\widetilde{T}_{\widetilde{\Omega}}}(\zeta,\eta) \right| g(\eta)\, d\lambda_\nu(\eta). \end{gathered} \tag{43}$$

In this case we can even state a stronger result. We consider a new operator, acting on functions in $\widetilde{\Omega}$ by

$$f \longmapsto \mathscr{Q} f(y) = \chi_{\mathbf{B}}(y) \int_{\widetilde{\Omega}\cap\mathbb{B}} \frac{f(u)}{\Delta(y+u)^{\nu-(n/r)}} d\lambda_\nu(u), \quad y \in \widetilde{\Omega}. \tag{44}$$

Here, with a slight abuse of notation, we still write $d\lambda_\nu$ for the measure $\Delta^{\nu-(2n/r)}(u)du$ in $\widetilde{\Omega}$ and $\mathbf{B}$ for the closed unit ball in $\mathbb{R}^n$. We write $L_\nu^{\widetilde{p},p}(\widetilde{T}_{\widetilde{\Omega}}) = L_y^p(\widetilde{\Omega}, \lambda_\nu; L_x^{\widetilde{p}}(\mathbb{R}^n))$, that is, the space with mixed norm given by

$$\|F\|_{L_\nu^{\widetilde{p},p}} = \left(\int_{\widetilde{\Omega}} \left[\int_{\mathbb{R}^n} |F(x+iy)|^{\widetilde{p}} dx \right]^{p/\widetilde{p}} d\lambda_\nu(y) \right)^{1/p}, \tag{45}$$

where $p, \widetilde{p} \in (0,\infty]$.

Proposition F (see [9]). Let $\nu > (2n/r)-1$ and $1 \leq q \leq p \leq \infty$. The following are equivalent:

(a) P_ν^+ is bounded from $L^p(\widetilde{D}, d\mu_\nu) \to L^q(\widetilde{D}, d\mu_\nu)$;

(b) $\mathscr{P}_\nu^{+,\mathbb{B}}$ is bounded from $L^p(\widetilde{T}_{\widetilde{\Omega}}, d\lambda_\nu) \to L^q(\widetilde{T}_{\widetilde{\Omega}}, d\lambda_\nu)$;

(c) $\mathscr{P}_\nu^{+,\mathbb{B}}$ is bounded from $L_\nu^{\widetilde{p},p}(\widetilde{T}_{\widetilde{\Omega}}) \to L_\nu^{\widetilde{q},q}(\widetilde{T}_{\widetilde{\Omega}})$ for all $1 \leq \widetilde{q} \leq \widetilde{p} \leq \infty$;

(d) $\mathscr{Q}$ is bounded from $L^p(\widetilde{\Omega}, d\lambda_\nu) \to L^q(\widetilde{\Omega}, d\lambda_\nu)$.

The Bergman projection P_0 for a bounded domain in $\mathbf{C}^n$, along with the associated Bergman kernel function, has proved fundamental for the theory of boundary behavior of holomohic mappings (see [27, 28] and their references).

It is well-known that on smooth bounded strongly pseudoconvex domains the Bergman projection preserves each Sobolev space W^r, $r \geq 0$. The same is true more generally for those smooth bounded weakly pseudoconvex domains which admit subelliptic estimates for the $\overline{\partial}$-Neumann problem (see [27, 28]).

By the mean value property for $n \geq 1$ we have for any bounded D domain $\mathbb{C}^n$

$$\begin{gathered} |f(z)| \leq C\|f\|_{A_0^p}^p \cdot (\operatorname{dist}(z,\partial D))^{-n}, \\ A_0^p = \left\{ f \in H(D) : \int_D |f(z)|^p d\delta(z) < \infty \right\}, \quad 0 < p < \infty, \end{gathered} \tag{46}$$

where $z \in D$ and $\delta(z)$ is Lebesgues measure on D and dist is Euclidean distance from z to the boundary of D.

This immediately pose a dist problem in all such very general D domains. We give one side estimates based on recent results on Bergman projection in Lie ball, bounded symmetric domains of tube type, Siegel domains of second type, and minimal bounded homogeneous domains.

We follow also our sharp recent result in unit disk. Note if bounded D domain has smooth C^2 boundary, then

$$|f(z)| \left[\operatorname{dist}(z,\partial D)^{(n+1)/p} \right] \leq C\|f\|_{A_0^p}. \tag{47}$$

A bounded domain in $\mathbb{C}^n$, $\Omega \subset \mathbb{C}^n$ is called smooth if there is a C^∞ defining function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\Omega = \{z : \rho < 0\}$ the boundary $b\Omega = \{z : \rho(z) = 0\}$ and the gradient of ρ does not vanish in $b\Omega$ (see [7, 12, 14, 23, 29]).

Inequality similar to (47) also is valid for weighted Bergman spaces (see [30]).

3. On Distance Function in Bergman Type Spaces in Certain Domains in Higher Dimension

In this section we provide main results of the paper. Based fully on preliminaries of previous section, the plan of this section is the following: first we formulate a result in the unit disk and then repeat arguments we provided in proof of that theorem in various situations in Bergman type spaces in Lie ball, bounded symmetric domains of tube type, Siegel domains of second type, and minimal bounded homogeneous domains. Since all proofs are short the repetitions of arguments which are needed in higher dimension will be omitted; sometimes sketches will be given.

Our results on Siegel domains and minimal homogeneous domains are sharp.

Let $\mathbf{U}$ be, as usual, the unit disk on the complex plane, and let $dm_2(z)$ be the normalized Lebesgue measure on $\mathbf{U}$. Let $H(\mathbf{U})$ be the space of all analytic functions on the unit disk $\mathbf{U}$. For $f \in H(\mathbf{U})$ and $f(z) = \sum_k a_k z^k$, define the fractional derivative of the function f as usual in the following manner:

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \quad \alpha \in \mathbb{R}. \tag{48}$$

We will write $Df(z)$ if $\alpha = 1$. Obviously, for all $\alpha \in \mathbb{R}$, $D^\alpha f \in H(\mathbf{U})$ if $f \in H(\mathbf{U})$.

For $k > s$, $0 < p, q \le \infty$, the weighted analytic Besov space $A_s^{q,p}(\mathbf{U})$ is the class of analytic functions satisfying

$$\|f\|_{A_s^{q,p}}^q = \int_0^1 \left(\int_{\mathbf{T}} \left| D^k f(r\xi) \right|^p |d\xi| \right)^{q/p} (1-r)^{(k-s)q-1} dr < \infty, \tag{49}$$

where $\mathbf{T} = \{\xi : |\xi| = 1\}$ is circle and $d\xi$ is the Lebesgue measure on the circle $\mathbf{T}$.

We denote by $A_s^q(\mathbf{U})$ the $A_s^{q,q}(\mathbf{U})$ analytic Besov spaces in the unit disk for all real s numbers. Note also that for $s < 0$ we have that these spaces are $A_{-sq-1}^q(\mathbf{U})$ Bergman spaces according to definition above for unit ball and we will use this notation below for all negative s numbers in Besov spaces.

It is well-known that $A_{-sq-1}^q(\mathbf{U}) \subset A_{-t}^\infty(\mathbf{U})$, $t = s - (1/q)$, $t < 0$, $s < 0$.

Let further $\Omega_{\varepsilon,-t}^k = \{z \in \mathbf{U} : |D^k f(z)|(1-|z|^2)^\varepsilon \ge -t\}$, $\varepsilon \ge 0$, $t < 0$, $\Omega_{\varepsilon,-t}^0 = \Omega_{\varepsilon,-t}$.

It is easy to note that, based on previous section results, the complete analogues of the embedding we just provided are valid also in Bergman type spaces in Lie ball, bounded symmetric domains of tube type, Siegel domains of second type, and minimal bounded homogeneous domains. We leave this easy task to readers. This allows posing a dist problem in each space we consider in this paper.

In the following theorem (see [1]) we calculate distances from a weighted Bloch class to Bergman spaces for $q \le 1$. We will see that almost each argument below is also valid not only in unit disk but based on preliminaries in previous section in Bergman type spaces in Lie ball, bounded symmetric domains of tube type, Siegel domains of second type, and minimal bounded homogeneous domains. So this theorem is very typical for us though it is known.

Theorem 4. *Let $0 < q \le 1$, $s < 0$, $t \le s - (1/q)$, $\beta > ((1-sq)/q) - 2$, and $\beta > -1 - t$. Let $f \in A_{-t}^\infty$. Then the following are equivalent:*

(a) $l_1 = dist_{A_{-t}^\infty}(f, A_{-sq-1}^q)$;

(b) $l_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{U}} (\int_{\Omega_{\varepsilon,-t}(f)} ((1-|w|)^{\beta+t}/|1-\overline{z}w|^{2+\beta}) dm_2(w))^q (1-|z|)^{-sq-1} dm_2(z) < \infty\}$.

Proof. First we show that $l_1 \le Cl_2$. For $\beta > -1-t$, we have

$$\begin{aligned} f(z) = C(\beta) \Bigg(& \int_{\mathbf{U} \setminus \Omega_{\varepsilon,-t}} \frac{f(w)(1-|w|)^\beta}{(1-\overline{w}z)^{\beta+2}} dm_2(w) \\ & + \int_{\Omega_{\varepsilon,-t}} \frac{f(w)(1-|w|)^\beta}{(1-\overline{w}z)^{\beta+2}} dm_2(w) \Bigg) \\ = {} & f_1(z) + f_2(z), \end{aligned} \tag{50}$$

where $C(\beta)$ is a well-known Bergman representation constant (see [31]).

For $t < 0$,

$$\begin{aligned} |f_1(z)| & \le C \int_{\mathbf{U} \setminus \Omega_{\varepsilon,-t}} \frac{|f(w)|(1-|w|)^\beta}{|1-\overline{w}z|^{\beta+2}} dm_2(w) \\ & \le C\varepsilon \int_{\mathbf{U}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}} dm_2(w) \\ & \le C\varepsilon \frac{1}{(1-|z|)^{-t}}. \end{aligned} \tag{51}$$

So $\sup_{z \in \mathbf{U}} |f_1(z)|(1-|z|)^{-t} < C\varepsilon$.

For $s < 0$, $t < 0$, we have

$$\begin{aligned} & \int_{\mathbf{U}} |f_2(z)|^q (1-|z|)^{-sq-1} dm_2(z) \\ & \le C \int_{\mathbf{U}} \left(\int_{\Omega_{\varepsilon,-t}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}} dm_2(w) \right)^q (1-|z|)^{-sq-1} dm_2(z) \\ & \le C. \end{aligned} \tag{52}$$

So we finally have

$$\operatorname{dist}_{A_{-t}^\infty}\left(f, A_{-sq-1}^q\right) \le C\|f - f_2\|_{A_{-t}^\infty} = C\|f_1\|_{A_{-t}^\infty} \le C\varepsilon. \tag{53}$$

It remains to prove that $l_2 \le l_1$. Let us assume that $l_1 < l_2$. Then we can find two numbers $\varepsilon, \varepsilon_1$ such that $\varepsilon > \varepsilon_1 > 0$, and a function $f_{\varepsilon_1} \in A_{-sq-1}^q$, $\|f - f_{\varepsilon_1}\|_{A_{-t}^\infty} \le \varepsilon_1$, and $\int_{\mathbf{U}} (\int_{\Omega_{\varepsilon,-t}} ((1-|w|)^{\beta+t}/|1-\overline{z}w|^{\beta+2}) dm_2(w))^q (1-|z|)^{-sq-1} dm_2(z) = \infty$. Hence as above we easily get from

$\|f - f_{\varepsilon_1}\|_{A^{\infty}_{-t}} \le \varepsilon_1$ that $(\varepsilon - \varepsilon_1)\chi_{\Omega_{\varepsilon,-t}(f)}(z)(1-|z|)^t \le C|f_{\varepsilon_1}(z)|$, and hence

$$\begin{aligned} M &= \int_{\mathbf{U}} \left(\int_{\mathbf{U}} \frac{\chi_{\Omega_{\varepsilon,-t}(f)}(z)(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}} dm_2(w) \right)^q \\ &\quad \times (1-|z|)^{-sq-1} dm_2(z) \\ &\le C \int_{\mathbf{U}} \left(\int_{\mathbf{U}} \frac{|f_{\varepsilon_1}(w)|(1-|w|)^{\beta}}{|1-\overline{w}z|^{\beta+2}} dm_2(w) \right)^q \\ &\quad \times (1-|z|)^{-sq-1} dm_2(z). \end{aligned} \tag{54}$$

Since for $q \le 1$, (see [31])

$$\begin{aligned} &\left(\int_{\mathbf{U}} \frac{|f_{\varepsilon_1}(w)|(1-|w|)^{\alpha}}{|1-\overline{w}z|^{t}} dm_2(w) \right)^q \\ &\quad \le C \int_{\mathbf{U}} \frac{|f_{\varepsilon_1}(w)|^q (1-|w|)^{\alpha q+q-2} dm_2(w)}{|1-\overline{w}z|^{tq}}, \end{aligned} \tag{55}$$

where $\alpha > (1-q)/q$, $t > 0$, $f_{\varepsilon_1} \in H(\mathbf{U})$, $z \in \mathbf{U}$, and

$$\int_{\mathbf{U}} \frac{(1-|z|)^{-sq-1}}{|1-\overline{w}z|^{q(\beta+2)}} dm_2(z) \le \frac{C}{(1-|w|)^{q(\beta+2)+sq-1}}, \tag{56}$$

where $s < 0$, $\beta > ((1-sq)/q) - 2$, $w \in \mathbf{U}$. we get

$$M \le C \int_{\mathbf{U}} |f_{\varepsilon_1}(z)|^q (1-|z|)^{-sq-1} dm_2(z), \tag{57}$$

So, we arrive at a contradiction. The theorem is proved. □

The following theorem is a version of Theorem 4 for the case $q > 1$.

Theorem 5. *Let $q > 1$, $s < 0$, $t \le s - (1/q)$, $\beta > (-1-sq)/q$, and $\beta > -1-t$. Let $f \in A^{\infty}_{-t}$. Then the following are equivalent:*

(a) $l_1 = dist_{A^{\infty}_{-t}}(f, A^q_{-sq-1})$;

(b) $\widehat{l}_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{U}} (\int_{\Omega_{\varepsilon,-t}(f)} ((1-|w|)^{\beta+t}/|1-\overline{z}w|^{2+\beta}) dm_2(w))^q (1-|z|)^{-sq-1} dm_2(z) < \infty\}$.

The proof of Theorem 5 is the same actually as the proof of Theorem 4. The only difference is the boundedness of Bergman type projection operator but with the positive Bergman kernel. This fact will be heavily used by us below.

Indeed the close inspection of the proof of Theorem 4 shows that the proof of Theorem 5 is the same as the proof of Theorem 4, but here we will use (58) (see below) instead of (55). For $\varepsilon > 0$, $q > 1$, $\beta > 0$, $\alpha > -1/q$,

$$\begin{aligned} &\left(\int_{\mathbf{U}} \frac{|f(z)|(1-|z|)^{\alpha}}{|1-\overline{w}z|^{\beta+2}} dm_2(z) \right)^q \\ &\le C \int_{\mathbf{U}} \frac{|f(z)|^q (1-|z|)^{\alpha q}}{|1-\overline{w}z|^{\beta q-\varepsilon q+2}} dm_2(z)(1-|w|)^{-\varepsilon q}, \quad w \in \mathbf{U}, \end{aligned} \tag{58}$$

which follows immediately from Hölder's inequality and (56).

Remark 6. Analytic A^{∞}_{α} spaces in unit disk are well-known in literature as so-called growth spaces (see, e.g., [31]). It can be shown easily that these spaces are Banach spaces. These spaces are playing a vital role in this paper and are embedded in Bergman spaces A^p_{β}, for large enough β (the same is valid in any bounded domain with C^2 boundary). Hence the representation (4) is valid also in the unit disk with β large enough index depending on α for all functions from such classes. The mentioned embedding is well-known and almost obvious and we leave the proof of it to interested readers. This fact is crucial for the proofs of Theorems 7 and 8 below and along with well-known Forelli-Rudin estimates, for to each domain we defined, (see [31]), actually serves as base of both proofs.

We formulate a general theorem for Siegel domains of second type in $\mathbb{C}^n$. Then we use the same ideas to formulate the same result in minimal bounded homogeneous domains in $\mathbb{C}^n$ using very recent advances of Yamaji (see [20, 22] and references there).

We formulate finally also some new one side estimates for distances based on projection theorems for symmetric domain of tube type, Lie ball, without proofs, since arguments are similar to unit disk case. Our results for Siegel domains of second type and minimal homogeneous domains are sharp.

Theorem 7. *Let D be Siegel domain of second type. Let*

$$r_j^0 = \max\left(\frac{n_i+2}{2(2d-q)_i}, -1 - \frac{n_i}{2(2d-q)_i} \right), \quad j = 1, \dots, l. \tag{59}$$

Let $f \in A^{\infty}_{1+r}$ and

$$N_{\widetilde{\varepsilon},r}(f) = \left\{ z \in D : |f(z)| b^{1+r}(z,z) > \widetilde{\varepsilon} \right\}, \tag{60}$$

where $\widetilde{\varepsilon}$ is a positive number. Then the following two quantities are equivalent $l_1 \asymp l_2$, where

$$l_1 = dist_{\widetilde{A}^{\infty}_{1+r}}\left(f, \widetilde{A}^{1,r}\right) \tag{61}$$

$$\begin{aligned} l_2 = \inf \Bigg\{ \widetilde{\varepsilon} > 0 : &\int_D \left(\int_{N_{\widetilde{\varepsilon},r}(f)} \widetilde{b}^{-k+1+r}(\tau,\tau) \left|\widetilde{b}(\tau,z)\right|^{k+1} d\widetilde{\nu}(\tau) \right) \\ &\times \widetilde{b}^{-r}(z,z)\, d\widetilde{\nu}(z) < \infty \Bigg\}, \end{aligned} \tag{62}$$

for all r and k so that, $r = (r_1, \dots, r_l)$, $k = (k_1, \dots, k_l)$, $r_j \in (r_0^j, \infty)$, and $k_j \in (k_0, \infty)$, $j = 1, \dots, l$ and for certain fixed vector $k_0 = (k_0^1, \dots, k_0^l)$, depending on r_j and on parameters of the Siegel D domains d_i, q_i, and n_i.

Proof of Theorem 7. We will follow the proof of Theorem 4. First we show that $l_1 \le C l_2$. Let $k \in \mathbb{R}^l$, $r \in \mathbb{R}^l$, $p \in \mathbb{R}_+$, $r_j \ge 0$, $j = 1, \dots, l$. Then $f(z) = C_{\varepsilon} \int_D f(u) b^{1+k}(z,u) b^{-k}(u) du$,

$f \in A^{\infty}_{1+r}$, $z \in D$, $k_i > m_i$ for the same large enough m_i set depending on parameters of Siegel domain, for all $f \in A^{1,r}(D)$.

Hence we have that $f = f_1 + f_2$

$$\begin{aligned} \left|f_1(z)\right| &\leq C \int_{D\setminus N_{\tilde{\varepsilon},r}} \left|f(u)\right| \left|b^{1+k}(z,u)\right| b^{-k}(u,u)\,du \\ &\leq C\varepsilon \int_D \left|b^{1+k}(z,w)\right| b^{-k+1+r}(u,u)\,d\nu(u) \\ &\leq C\varepsilon b(z,z)^{1+r}, \quad z \in D. \end{aligned} \tag{63}$$

Also, we have that

$$\begin{aligned} &\int_D \left|f_2(z)\right| b^{-r}(z,z)\,d\nu(z) \\ &\leq \int_D \left(\int_{N_{\tilde{\varepsilon},r}} b^{-k+1+r}(\tau,\tau)\left|b(\tau,z)\right|^{k+1} d\nu(\tau)\right) b^{-r}(z,z)\,d\nu(z) \\ &\leq C. \end{aligned} \tag{64}$$

Note then we used that for

$$\begin{gathered} k_j - 1 - r_j > \frac{n_i + 2}{2(2d-q)_i} \\ k_j - 1 - r_j - \frac{n_i}{2(2d-q)_i} < k_j, \quad j = 1, \dots, l, \\ \int_D \left|b^{1+k}(z,u)\right| b^{-k+1+r}(u,u)\,d\nu(u) \leq Cb(z,z)^{1+r}, \quad z \in D. \end{gathered} \tag{65}$$

So $\operatorname{dist}_{A^{\infty}_{1+r}}(f, A^{1,r}) \leq C \parallel f - f_2 \parallel_{A^{\infty}_{1+r}} = C \parallel f_1 \parallel_{A^{\infty}_{1+r}} \leq C\varepsilon$.

We show the reverse implication now following again unit disk case arguments (see Theorems 4 and 5). We have the following. Let us assume that $l_1 < l_2$. Then we can find two numbers $\varepsilon, \varepsilon_1$ such that $\varepsilon > \varepsilon_1 > 0$ and a function $f_{\varepsilon_1} \in A^{1,r}$, $\|f - f_{\varepsilon_1}\|_{A^{1,r}} \leq \varepsilon_1$ and hence

$$(\varepsilon - \varepsilon_1)\,\chi_{N_\varepsilon(f)}(z)\,b(z,z)^{1+r} \leq C\left|f_{\varepsilon_1}(z)\right| \tag{66}$$

and hence

$$\begin{aligned} &\int_D \left(\int_D \chi_{N_\varepsilon(f)}(z)\,b(z,z)^{1+r-k}\left|b(\tau,z)\right|^{k+1} d\nu(z)\right) \\ &\quad \times b^{-r}(\tau,\tau)\,d\nu(\tau) \\ &\quad \leq C\int_D \left(\int_D \left|f_{\varepsilon_1}(z)\right| b(z,z)^{-k}\left|b(\tau,z)\right|^{k+1} d\nu(z)\right) \\ &\qquad \times b^{-r}(\tau,\tau)\,d\nu(\tau) \\ &\quad \leq C\int_D \left|f_{\varepsilon_1}(z)\right| b(z,z)^{-r} d\nu(z). \end{aligned} \tag{67}$$

We used that (see [5, 8])

$$\begin{gathered} \int_D \left|b(\tau,z)\right|^{k+1} b^{-r}(\tau,\tau)\,d\nu(\tau) \leq C(b(z,z))^{k-r}, \quad z \in D \\ r_j > \frac{n_i + 2}{2(2d-q)_i}, \quad i = 1, \dots, l, \\ k_j > r_j - \frac{n_i}{2(2d-q)_i}, \quad i = 1, \dots, l. \end{gathered} \tag{68}$$

These estimates give a lower estimate for $k_0 = (k_0^1, \dots, k_0^l)$.

Theorem 7 is proved. □

For $p > 1$ and $\alpha > -1$, denote by $A^p_\alpha(D)$ the space of all functions f holomorphic in D and satisfying the condition

$$\int_D \left|f(w)\right|^p \operatorname{dist}(z,\partial D)^\alpha d\nu(w) < +\infty, \tag{69}$$

where $d\nu$ is the Lebesgue measure in D domain.

Let K_α denote reproducing weighted Bergman kernel for weighted Bergman space A^2_α (see [11, 12])

$$K_\alpha(z,w) = K(z,w)^{1+\alpha}, \tag{70}$$

where $K(z,w)$ is a Bergman kernel for A^2_0. Based on preliminaries of previous section on Bergman type spaces in minimal homogeneous domains and arguments of proof of Theorems 4 and 5 and comments related to them, we can formulate the following result.

Note that it is known that there exists a constant $\alpha_{\min}$ such that A^2_α is nontrivial if and only if $\alpha > \alpha_{\min}$, (see [11, 12]). From now, we consider nontrivial weighted Bergman spaces.

Theorem 8. *Assume that $\alpha > \alpha_{\min}$ and let $\mathcal{U}$ be bounded minimal homogeneous domains with C^2 boundary. Let $G_{\varepsilon,\alpha}(f) = \{z \in \mathcal{U} : |f(z)||K_\alpha(z,z)|^{-1/2} > \varepsilon\}$. Then for $f \in A^\infty_\alpha$ we have that $l_1 \asymp l_2$, where*

$$\begin{aligned} l_1 &= dist_{A^\infty_\alpha}\left(f, A^2_\alpha\right), \\ l_2 &= \inf\left\{\varepsilon > 0 : \int_{\mathcal{U}} \left(\int_{G_{\varepsilon,\alpha}} K(z,z)^{(1-\alpha)/2}\left|K_\alpha(z,a)\right| d\nu(z)\right)^2 \right. \\ &\qquad \left. \times K(a,a)^{-\alpha} d\nu(a) < \infty\right\}, \end{aligned} \tag{71}$$

for all $\alpha > \alpha_0$ for some fixed α_0.

We add only the full sketch of proof of this theorem since it is quite similar to proofs of previous theorems. We consider the operator $P^+_{\mathcal{U}}$ defined by

$$P^+_{\mathcal{U}} g(z) = \int_{\mathcal{U}} \left|K_{\mathcal{U}}(z,w)\right| g(w)\,d\nu(w), \quad g \in L^2(\mathcal{U}, d\nu). \tag{72}$$

$P^+_{\mathcal{U}}$ is operator $\mathcal{U}$ on $L^2(\mathcal{U}, d\nu)$. The fact that Bergman type projection $P^+_{\mathcal{U}}$ with positive Bergman kernel is a bounded operator in L^2_α spaces (weighted Lebegue spaces) can be seen in recent paper [11]. It is well-known also, since A^2_α is a Hilbert space that the Bergman reproducing formula for all functions from these spaces is valid and since our domain is bounded, this space for large enough α contains A^∞_β for any β. So the Bergman representation formula for each function taken from A^∞_β is also valid with large enough α. These two facts along with Forelly-Rudin type estimates for these domains (see [12]) complete the proof.

All other results of this paper are one side estimates for distance function and are fully based on results on boundedness of Bergman type projection with positive Bergman kernel as we have seen in unit disk case in Bergman type spaces, but over specific domain which we mentioned above in previous section as separate assertions.

Based on preliminaries of previous section on Bergman type spaces in Lie ball and arguments of proof of Theorems 4 and 5 and comments related to them, we can formulate the following result.

Theorem 9. *Let D be Lie ball with C^2 boundary. Let $p \in ((2n-2)/n, (2n-2)/n-2)$, $n > 2$, $f \in A^\infty_{n/p}$. Let $D_t = \{z : |f(z)| dist(z, \partial D)^t \geq \varepsilon\}$, $t = n/p$. Then*

$$\begin{aligned} & dist_{A^\infty_{n/p}}\left(f, A^p_0\right) \\ & \geq C \inf\left\{\varepsilon > 0 : \int_D \left(\int_D \chi_{D_t}(z)\, dist(z, \partial D)^t \right.\right. \\ & \qquad \left.\left. \times B_D(z, w) d\nu(z)\right)^p d\nu(w) < \infty\right\}. \end{aligned} \tag{73}$$

We omit details of the proof referring to unit disk case.

Based on preliminaries of previous section on Bergman type spaces in bounded symmetric domains of tube type and arguments of proof of Theorems 4 and 5 and comments related to them, we can formulate the following result.

Theorem 10. *Let D be bounded symmetric domain of tube type with C^2 boundary. Let $p \in (1, \infty)$, $n > 2$, $f \in A^\infty_{n/p}$. Let $D^1_t = \{z : |f(z)| dist(z, \partial D)^t \geq \varepsilon\}$, $t = n/p$. Then*

$$\begin{aligned} & dist_{A^\infty_{n/p}}\left(f, A^p_0\right) \\ & \geq C \inf\left\{\varepsilon > 0 : \int_D \left(\int_D \chi_{D^1_t}(z)\, dist(z, \partial D)^t B^D_\nu \right.\right. \\ & \qquad \left.\left. \times (z, w)\, d\nu(z)\right)^p d\nu(w) < \infty\right\}. \end{aligned} \tag{74}$$

Remark 11. It is known that (see [30]) if $\alpha > -1$ and $p \in (0, \infty)$ and if D is general bounded domain with C^2 boundary then

$$\text{dist}(z, \partial D)^{(n+\alpha+1)/p} |f(z)| \leq C\|f\|_{A^p_\alpha}. \tag{75}$$

This allows posing a dist problem for weighted Bergman spaces in such type domains and based on preliminaries of previous section to formulate Theorem 9 even for weighted Bergman spaces. We leave this to readers.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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