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Coxeter Groups as Automorphism Groups of Solid Transitive 3-simplex Tilings

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Abstract. In the papers of I.K. Zhuk, then more completely of E. Molnár, I. Prok, J. Szirmai all simplicial 3-tilings have been classified, where a symmetry group acts transitively on the simplex tiles. The involved spaces depends on some rotational order parameters. When a vertex of a such simplex lies out of the absolute, e.g. in hyperbolic space H^3, then truncation with its polar plane gives a truncated simplex or simply, trunc-simplex.

Looking for symmetries of these tilings by simplex or trunc-simplex domains, with their side face pairings, it is possible to find all their group extensions, especially Coxeter's reflection groups, if they exist. So here, connections between isometry groups and their supergroups is given by expressing the generators and the corresponding parameters. There are investigated simplices in families F3, F4, F6 and appropriate series of trunc-simplices. In all cases the Coxeter groups are the maximal ones.

1. Introduction

The isometry groups, acting discontinuously on the hyperbolic 3-space with compact fundamental domain, are called hyperbolic space groups. One possibility to describe them is to look for their fundamental domains. Face pairing identifications of a given polyhedron give us generators and relations for a space group by Poincaré Theorem [1], [3], [7].

The simplest fundamental domains are simplices and truncated simplices by polar planes of vertices when they lie out of the absolute. There are 64 combinatorially different face pairings of fundamental simplices [17], [6], furthermore 35 solid transitive non-fundamental simplex identifications [6]. I. K. Zhuk [17] has classified Euclidean and hyperbolic fundamental simplices of finite volume up to congruence. Some completing cases are discussed in [2], [5], [11], [12], [13], [14], [15], [16]. An algorithmic procedure is given by E. Molnár and I. Prok [5]. In [6], [8] and [9] the authors summarize all these results, arranging identified simplices into 32 families. Each of them is characterized by the so-called maximal series of simplex tilings. Besides spherical, Euclidean, hyperbolic realizations there exist also other metric realizations in 3-dimensional simply connected homogeneous Riemannian spaces, moreover, metrically non-realizable topological simplex tilings occur as well [4].

This paper is a continuation of [14] and the presented results are as follows:

Main result. *Investigating symmetries of the series of simplices in families F3, F4, F6 and corresponding series of trunc-simplices, it is given relationship between their isometry groups and possible supergroups. The maximal*

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supergroups are always the Coxeter reflection groups, whose realizations can be described by the usual machinery, illustrated here only for family F6.

Moreover, there are also constructed some new series of fundamental domains, by truncating simplices and half-simplices. The results will be given in Sections 3, 4 and in tables. Notations used here for simplices and families of simplices are introduced in [6], [8], [9], while the investigated trunc-simplices are introduced in [14].

2. Maximal tilings and principles of truncating simplices

2.1 Let $(\mathcal{T}, \Gamma)$ denote a *face-to-face tiling* $\mathcal{T}$ by topological simplices $T_1, T_2, \ldots$ as a usual incidence structure of 0-, 1-, 2-, 3-dimensional constituents in a simply connected topological 3-space $(\mathcal{X}^3, G)$ with a group $\Gamma < G$ that acts on $\mathcal{T}$ by topological mapping of $\mathcal{X}^3$, preserving the incidences.

Let Γ be *tile-transitive* on $\mathcal{T}$, i.e. for any two 3-tiles $T_1, T_2 \in \mathcal{T}$ there exists (at least one) $\gamma \in \Gamma$ such that

$$\gamma : \mathcal{T} \to \mathcal{T}, \quad T_1 \mapsto T_2 := T_1^{\gamma}.$$

Two tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are called *combinatorially* (topologically) *equivariant*, and lie in the same *equivariance class*, iff there is a bijective incidence preserving (topological) mapping

$$\varphi : \mathcal{T} \to \mathcal{T}', \quad T \mapsto T' := T^{\varphi} \text{ such that } \Gamma' = \varphi^{-1}\Gamma\varphi.$$

$(\mathcal{T}, \Gamma)$ is a *maximal tiling* iff $\Gamma = \mathrm{Aut}\mathcal{T}$ is a (maximal) *group of automorphisms* of $\mathcal{T}$. A maximal tiling $(\mathcal{T}^* = \mathcal{T}, \Gamma^* = \mathrm{Aut}\mathcal{T})$ represents a *family of tilings* $(\mathcal{T}, \Gamma)$ such that there is a bijection (topological mapping) above

$$\varphi : \mathcal{T} \to \mathcal{T}^*, \text{ with } \varphi^{-1}\Gamma\varphi \leq \Gamma^* = \mathrm{Aut}\mathcal{T}.$$

2.2 When the rotational order parameters $(u, v, \ldots)$ are such that the simplex T_i (used enumeration for the simplices is the same as in [6, 8, 9]) is hyperbolic and vertices in an equivalence class t are out of the absolute, then it is possible to truncate the simplex by polar planes of these vertices. We get the trunc-simplex of finite volume (possibly with 8 faces, as octahedron) denoted by O_i^t. Dihedral angles around the new edges are $\pi/2$. It means there are four congruent trunc-simplices around such an edge in the fundamental space filling. We can equip O_i^t with additional pairings of the new triangular faces (trunc-faces). Then O_i^t will be a fundamental domain for the new group $G_j(O_i^t)$ which will be a supergroup of Γ_i, for each value of edge parameters. The trivial group extension is always possible with plane reflections in polar planes of the outer vertices. Let us denote this group by $G_1(O_i^t)$.

If there is a further possibility to equip the new triangular faces to outer vertices in class t, represented by A_k, with face pairing isometries, then the new additional face pairings of O_i^t have to satisfy the following criteria. The polar plane of A_k and so the stabilizer $\Gamma(A_k)$ will be invariant under these new transformations and exchange the half-spaces obtained by the polar plane. Thus, the fundamental domain P_{A_k} is divided into two parts, and the new stabilizer of the polar plane will be a supergroup for $\Gamma(A_k)$ of index two. Inner symmetries of the P_{A_k}-tiling give us the idea how to introduce the new generators.

Note that truncations of vertices in different equivalence classes can independently be combined, even it is possible to truncate vertices in a class and leave others without truncating.

3. Supergroups of the simplex and trunc-simplex tilings

Here will be discussed the supergroups of simplicial fundamental domains from the families F6, F3, F4, and the corresponding trunc-simplicial domains. In all cases Coxeter groups will be the maximal ones. Coxeter groups will be the maximal supergroups of the simplicial (and related trunc-simplicial) domains in the family F1, and also in all other families for special cases of parameters [2, 9, 11].

Family F6. This family is represented by its maximal group ${}^m_2\Gamma_1(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$, $(2 \le \bar{u}, 2 \le \bar{v} \le \bar{w}, 3 \le \bar{x})$ with minimal fundamental domain – a simplex $T_{F6} : A_0A_{01}A_2A_3$ (Fig.1). This domain is also described in [9], by D-symbols and Schlegel diagram. The group is special case of the reflection group $\Gamma_1(2a, 2b, 2c, 2d, 2e, 2f)$, where for the parameters hold $2a = \bar{x}$, $b = \bar{w}$, $c = 2$, $d = \bar{u}$ $e = 2$,$f = \bar{v}$ ([9]).

When parameters are such that $\frac{1}{\bar{u}} + \frac{1}{\bar{v}} + \frac{1}{\bar{w}} < 1$ vertices A_0, A_1 are outer. Also if $\frac{1}{2} + \frac{1}{\bar{v}} + \frac{1}{\bar{x}} < 1$, or $\frac{1}{2} + \frac{1}{\bar{w}} + \frac{1}{\bar{x}} < 1$, vertices A_2 and A_3 are outer and we can truncate them by polar planes of these vertices, respectively.

Vertex domains $F^0(A_0)$, $F^0(A_2)$ and $F^0(A_3)$ shown in Fig.1 are such that they haven't more symmetries (keeping the adjacencies, i.e. the types of the side lines are invariant), so the additional trunc-side pairings (Fig.2) are trivial by reflections $\bar{m}_i$, $(i = 0, 2, 3)$. It means, the obtained trunc-simplex is fundamental domain of the new Coxeter group with Coxeter diagram given in Fig.2. Vertices of this diagram indicate the reflection side planes. The edges marked e.g. with $\bar{u}$ means an angle $\frac{\pi}{\bar{u}}$ between the corresponding reflection planes. Not connected vertices means orthogonal planes. Vertices connected with dashed lines indicate not intersecting (and not parallel, with a common perpendicular line to both) planes. Similar conventions will be applied later on as well.

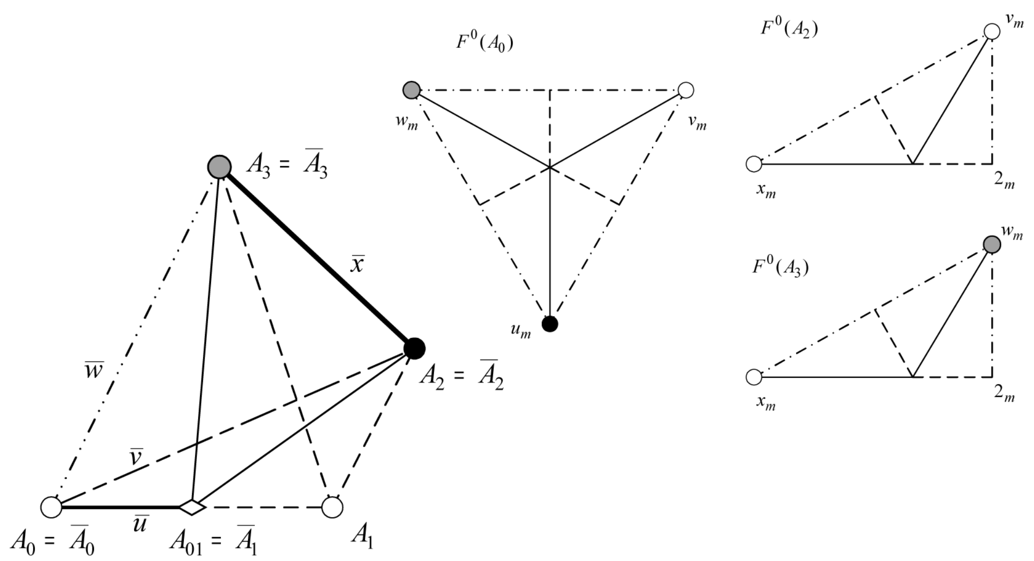

Figure 1: F6: Simplex T_{F6}; $F^0(A_i)$, $(i = 0, 2, 3)$

In family F6 there are four face paired simplices, shown in Fig.3,4. Possible trunc-simplices for them have been described in [14].

It would be obvious that groups for these four fundamental simplices are subgroups of ${}^m_2\Gamma_1(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$ (relationship between parameters are given in table) if we halve those simplices with plane α through edge A_2A_3 and midpoint $M = A_{01}$ of A_0A_1 (Fig. 5). In all cases we equip the new simplex $A_0A_2A_3M$ with face pairing by reflections

$$m_\alpha : \begin{pmatrix} M & A_2 & A_3 \\ M & A_2 & A_3 \end{pmatrix}, m_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}, m_2 : \begin{pmatrix} A_0 & M & A_3 \\ A_0 & M & A_3 \end{pmatrix}, m_3 : \begin{pmatrix} A_0 & M & A_2 \\ A_0 & M & A_2 \end{pmatrix}.$$

Since it is possible to express generators of each starting group by generators of group ${}^m_2\Gamma_1(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$, the last one is a supergroup for appropriate parameters.

Similarly, after halving the corresponding trunc-simplex with plane α the supergroup of $G_j(O_i^t)$ is the Coxeter group

$$\Gamma(O, 2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x}) = (m_\alpha, m_1, m_2, m_3, \bar{m}_0, \bar{m}_2, \bar{m}_3 - m_\alpha^2 = m_1^2 = m_2^2 = m_3^2 = (\bar{m}_0)^2 =$$

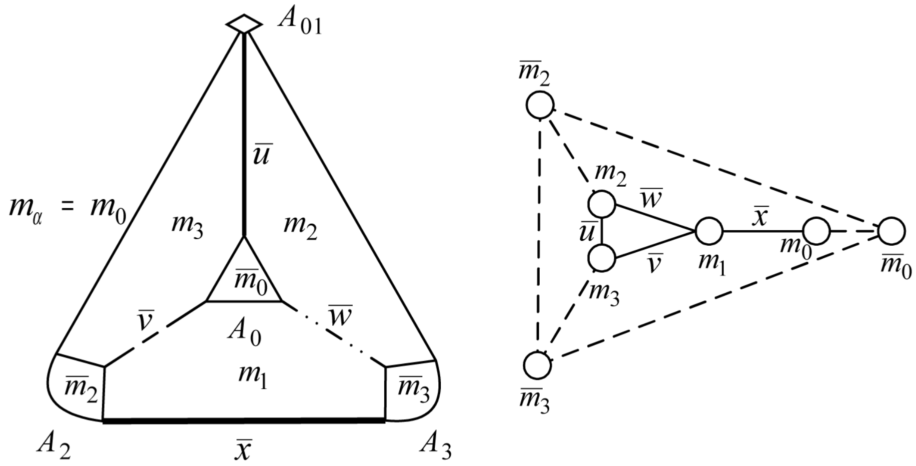

Figure 2: F6: trunc-simplex and its Coxeter diagram

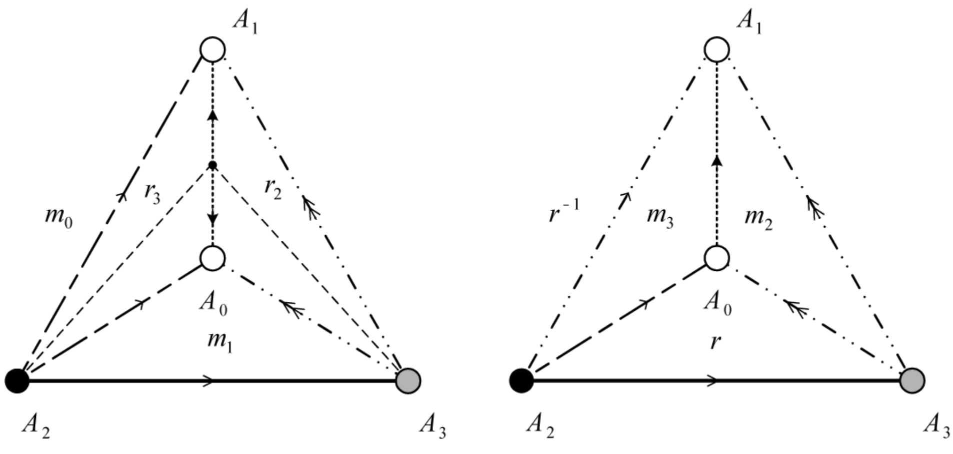

Figure 3: Simplices T_6 and T_{20} obtained by gluing two copies of T_{F6}

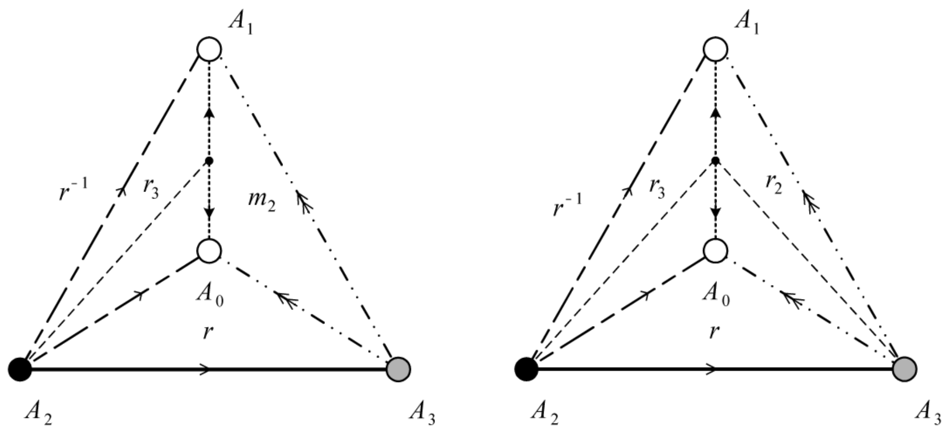

Figure 4: Simplices T_{24} and T_{35} obtained by gluing two copies of T_{F6}

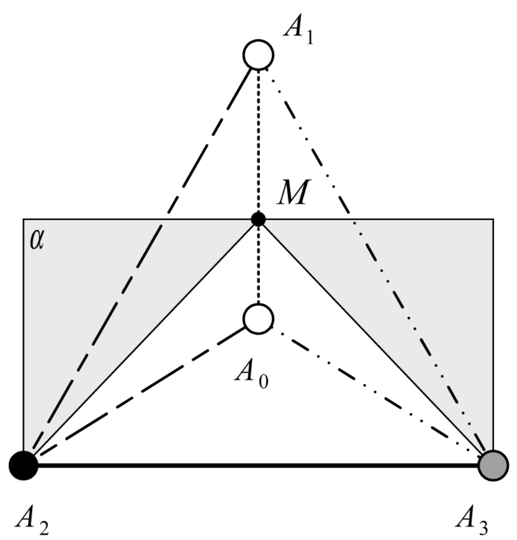

Figure 5: Halving a simplex from F6 with its symmetry plane α

$$= (\bar{m}_2)^2 = (\bar{m}_3)^2 = (m_\alpha m_2)^2 = (m_\alpha m_3)^2 = (m_2 m_3)^{\bar{u}} = (m_3 m_1)^{\bar{v}} = (m_2 m_1)^{\bar{w}} =$$

$$= (m_\alpha m_1)^{\bar{x}} = (\bar{m}_0 m_1)^2 = (\bar{m}_0 m_2)^2 = (\bar{m}_0 m_3)^2 = (\bar{m}_2 m_\alpha)^2 = (\bar{m}_2 m_1)^2 =$$

$$= (\bar{m}_2 m_3)^2 = (\bar{m}_3 m_\alpha)^2 = (\bar{m}_3 m_1)^2 = (\bar{m}_3 m_2)^2 = 1,$$

$$\frac{1}{\bar{u}} + \frac{1}{\bar{v}} + \frac{1}{\bar{w}} < 1,\ \frac{1}{2} + \frac{1}{\bar{v}} + \frac{1}{\bar{x}} < 1,\ \frac{1}{2} + \frac{1}{\bar{w}} + \frac{1}{\bar{x}} < 1).$$

Here it is assumed that all vertices A_0, A_2, A_3 are truncated, although it is possible that e.g. A_0 is not an outer vertex, since $\frac{1}{\bar{u}} + \frac{1}{\bar{v}} + \frac{1}{\bar{w}} \geq 1$. Note that the vertex M is always proper.

In Table 1 for each simplex from this family is given:

1. Generators for the simplex group Γ_i ($i = 6, 20, 24, 35$), with fundamental simplex T_i.
2. Relationship between parameters for isometries of Γ_i and their Coxeter's supergroup ${}^{m}_{2}\Gamma_1(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$.
3. Additional generators (pairing new triangular faces) for groups of truncated simplex O_i ($i = 6, 20, 24, 35$) given separately for each of the vertex classes $l : \{A_0, A_1\}$, $m : \{A_2\}$, $n : \{A_3\}$. If there are more possibilities, the trivial one is denoted by **I**. Vertices of trunc-face of vertex A_i are denoted by T_i^j, where $T_i^j \in A_i A_j$.

Table 1: Family 6

1. $\mathbf{T}_6$:$\Gamma_6(2u, 4v, 4w, 2x)$ m_0: $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}$; m_1: $\begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}$; r_2: $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}$; r_3: $\begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $u = \bar{u}, 2v = \bar{v}, 2w = \bar{w}, 2x = \bar{x}$; $r_2 = m_\alpha m_2, r_3 = m_\alpha m_3$
3. l) **I**: $\bar{m}_1$, $\bar{m}_0$; **II**: s: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^2 & T_1^3 \end{pmatrix}$ m) **I**: $\bar{m}_2$; **II**: h_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_2^1 & T_2^0 & T_2^3 \end{pmatrix}$ n) **I**: $\bar{m}_3$; **II**: h_3: $\begin{pmatrix} T_3^0 & T_3^1 & T_3^2 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$
1. $\mathbf{T}_{20}$:$\Gamma_{20}(2u, 4v, 4w, x)$ r : $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}$; m_2 : $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_3 \end{pmatrix}$; m_3 : $\begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_1 & A_2 \end{pmatrix}$
2. $u = \bar{u}, 2v = \bar{v}, 2w = \bar{w}, x = \bar{x}$; $r = m_\alpha m_1$
3. l) **I**: $\bar{m}_1$, $\bar{m}_0$; **II**: s: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^2 & T_1^3 \end{pmatrix}$ m) **I**: $\bar{m}_2$; **II**: h_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_2^1 & T_2^0 & T_2^3 \end{pmatrix}$ n) **I**: $\bar{m}_3$; **II**: h_3: $\begin{pmatrix} T_3^0 & T_3^1 & T_3^2 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$
1. $\mathbf{T}_{24}$:$\Gamma_{24}(4u, 2v, 4w, x)$ r : $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}$; m_2 : $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_3 \end{pmatrix}$; r_3 : $\begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $2u = \bar{u}, v = \bar{v}, 2w = \bar{w}, x = \bar{x}$; $r_3 = m_\alpha m_3, r = m_\alpha m_1$
3. l) **I**: $\bar{m}_1$, $\bar{m}_0$; **II**: s: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^2 & T_1^3 \end{pmatrix}$ m) **I**: $\bar{m}_2$; **II**: h_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_2^1 & T_2^0 & T_2^3 \end{pmatrix}$ n) **I**: $\bar{m}_3$; **II**: h_3: $\begin{pmatrix} T_3^0 & T_3^1 & T_3^2 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$

1. $\mathbf{T}_{35}$:$\Gamma_{35}(2u, 2v, 2w, x)$ $r : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}; r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}; r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $u = \bar{u}, v = \bar{v}, w = \bar{w}, x = \bar{x}; r_2 = m_\alpha m_2, r_3 = m_\alpha m_3, r = m_\alpha m_1$
3. $l)$ **I**: $\bar{m}_1$, $\bar{m}_0$; **II**: s: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^2 & T_1^3 \end{pmatrix}$ $m)$ **I**: $\bar{m}_2$; **II**: h_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_2^1 & T_2^0 & T_2^3 \end{pmatrix}$ $n)$ **I**: $\bar{m}_3$; **II**: h_3: $\begin{pmatrix} T_3^0 & T_3^1 & T_3^2 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$

Family F3. Family is represented by its maximal group ${}^{3m}_{6}\Gamma(2\bar{u}, \bar{v})$, $(2 \leq \bar{u}, 3 \leq \bar{v})$, which is a Coxeter group [9].

If the parameters are such that $\frac{1}{2} + \frac{1}{\bar{u}} + \frac{1}{\bar{v}} < 1$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{\bar{v}} < 1$, then vertices A_0 and A_2 respectively, are outer and we can truncate them by polar planes of these vertices. Since vertex domains $F^0(A_0)$ and $F^0(A_2)$ (Fig.6) haven't more symmetries, the additional trunc-side pairings are trivial by reflections $\bar{m}_0$ and $\bar{m}_2$ (Fig.6). So, such trunc-simplex is the fundamental domain of the new Coxeter group with Coxeter diagram given in (Fig.6).

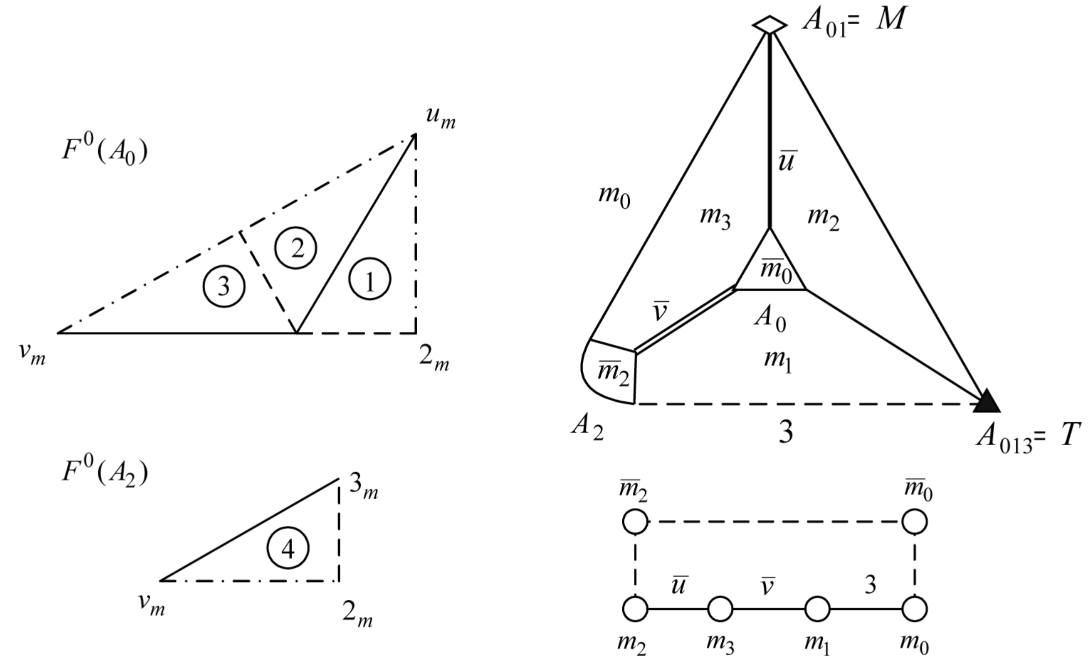

Figure 6: F3: $F^0(A_i)$, $(i = 0, 2)$; trunc-simplex; Coxeter diagram

The simplex group $\Gamma_{33}(12u, 6v)$ is subgroup of the former maximal group iff $6u = \bar{u}$ and $6v = \bar{v}$. That would be obvious after splitting the fundamental simplex T_{33} which has the generators

$$m_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}, r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}, z : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_1 & A_2 & A_0 \end{pmatrix}$$

into three congruent parts, we can obtain the supergroup of the starting one. The new fundamental simplex $A_1A_2A_3T$, where T is the (formal) barycenter of face $A_0A_1A_3$ with face pairing isometries

$$m_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}, \widehat{r_2} : \begin{pmatrix} A_1 & A_3 & T \\ A_3 & A_1 & T \end{pmatrix}, r : \begin{pmatrix} A_3 & A_2 & T \\ A_1 & A_2 & T \end{pmatrix}.$$

is simplex denoted by T_{24}. It means that the new group and also the starting one are subgroups of the Coxeter group ${}^{3m}_{6}\Gamma$. The old generators, expressed by the new ones, are

$$r_2 = r^{-1}\widehat{r_2}r, z = rm_0r.$$

Similar splitting of the trunc-simplex O_{33} (for each variant of paired faces) provides the fundamental domain of the supergroup $G_1(O_{24})$. So, again the Coxeter groups ${}^{3m}_{6}\Gamma$ and $G(O)$ are supergroups of Γ_{33} and $G_j(O_{33})$ $(j = 1, 2)$.

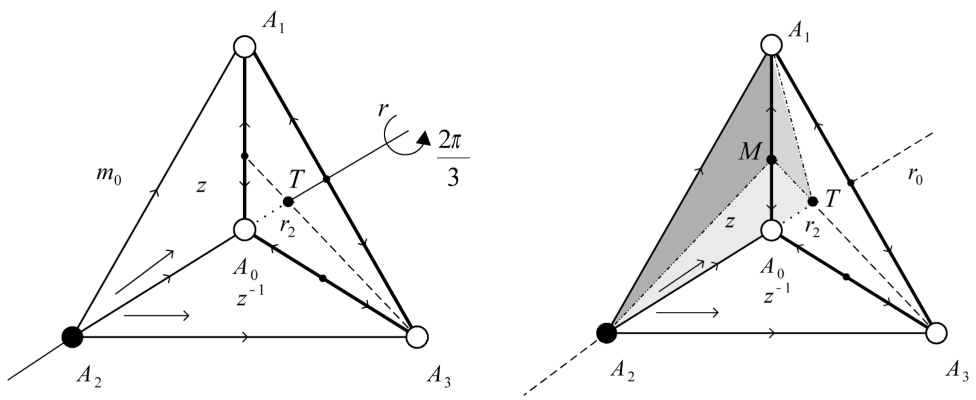

Figure 7: Symmetries of simplices T_{33}, T_{42} from F3, according to face pairings

Although another simplex T_{42} from family 3 has the same symmetries as T_{33}, the faces of the same splitting is not possible to equip with pairing isometries agreeing with those of T_{42}, since its isometry group $\Gamma_{42}(6u, 3v)$ has generators

$$r_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_3 & A_2 & A_1 \end{pmatrix}, r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}, z : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_1 & A_2 & A_0 \end{pmatrix}$$

A single one supergroup of isometry group with fundamental domain T_{42}, is Coxeter group ${}^{3m}_{6}\Gamma(2\bar{u}, \bar{v})$, $3u = \bar{u}$, $3v = \bar{v}$ with fundamental simplex e.g. A_2A_1MT where M is midpoint of edge A_0A_1.

Also, the only supergroup of $G_j(O_{42})$ $(j = 1, 2)$ is $G(O)$.

Family F4. Family F4 is represented by ${}^{mm2}_{4}\Gamma_1(\bar{u}, 2\bar{v}, \bar{w})$ $(3 \le \bar{u} \le \bar{w}, 2 \le \bar{v})$, the Coxeter group.

Vertices A_0, A_1 are outer if $\frac{1}{2} + \frac{1}{\bar{u}} + \frac{1}{\bar{v}} < 1$ and vertices A_2, A_3 are such that if $\frac{1}{2} + \frac{1}{\bar{u}} + \frac{1}{\bar{w}} < 1$. Then it is possible to truncate these vertices. Since there are no symmetries of $F^0(A_0)$ and $F^0(A_2)$ (Fig.8), the additional trunc-side pairings are trivial and new trunc-simplex is fundamental domain of the new Coxeter group with Coxeter diagram given in Fig.8.

Each of the six simplices (Fig. 9, 10, 11), from family F4, is possible to halve either with plane α as before, or with plane β through edge A_0A_1 and mid point N of A_2A_3 (Fig. 12). It is also possible to halve with both of them at the same time. The new faces of simplices $A_0A_2A_3M$ and $A_0A_1A_2N$ are paired with either plane reflection m_α and m_β, or with half-turn h around the axis MN. If h is used as a face-pairing of face MA_2A_3 or NA_0A_1, let us denote it by h_α or h_β, respectively. In the case of halving with α and β at the same time, a

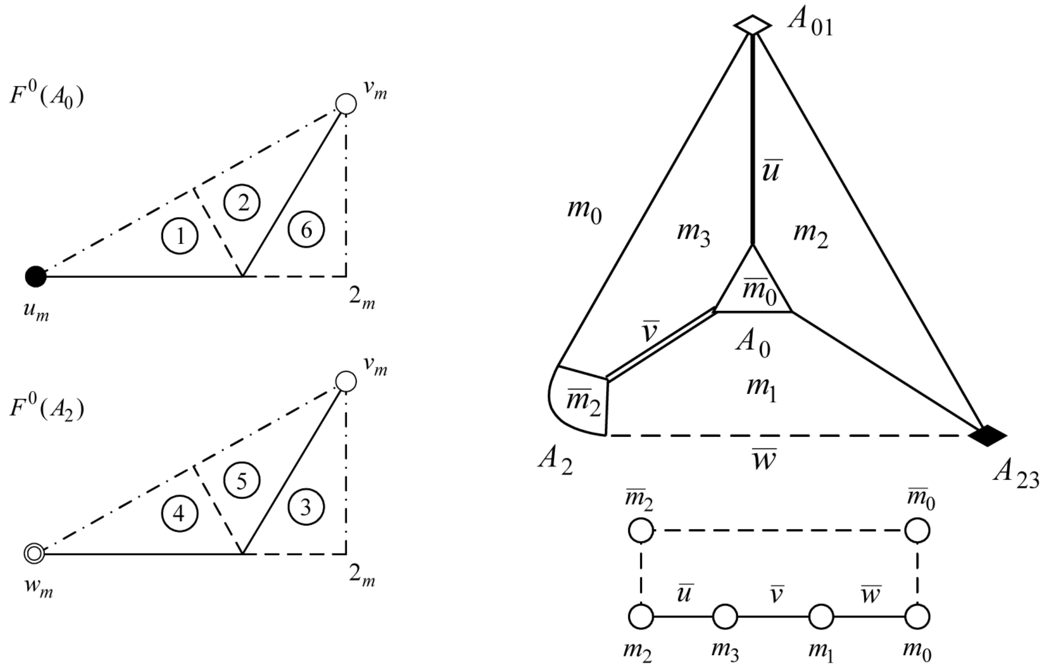

Figure 8: F4: $F^0(A_i)$, $(i = 0, 2)$; trunc-simplex; Coxeter diagram

new, quarter simplex A_0A_2MN, with plane reflections $m_1, m_3, m_\alpha, m_\beta$ pairing the faces, provides the Coxeter group ${}^{mm2}_{\ 4}\Gamma_1(\bar{u}, 2\bar{v}, \bar{w})$ as a supergroup of the starting ones.

When simplices T_{10} and T_{28} are divided by β it appears simplex T_2, with the group $\Gamma_2(4a, 4b, 2c, 2d, 2e)$ (by [9], $2a = u$, $b = 2v$, $c = 2w$, $e = d = 2$), from family 14. Face pairings of this simplex are:

$$m_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}, r_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}, m_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_3 \end{pmatrix}, m_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_1 & A_2 \end{pmatrix}.$$

The group is maximal in general case, but here it is $d = e$ and so, there are more symmetries (by plane α). Such new group Γ_2, has group ${}^{mm2}_{\ 4}\Gamma_1(\bar{u}, 2\bar{v}, \bar{w})$ as a supergroup.

In the case of simplices T_{17} and T_{38}, with groups $\Gamma_{17}(2u, 4v, 2w)$ and $\Gamma_{38}(2u, 4v, 2w)$, it is possible to obtain (by notations in [6], [8], [9], [14]) ${}^2_2\Gamma_4(\hat{u}, 2\hat{v}, \hat{w})$, with $2\bar{u} = \hat{u}$, $2\bar{v} = \hat{v}$, $2\bar{w} = \hat{w}$ and ${}^2_2\Gamma_3(2\hat{u}, 4\hat{v}, \hat{w})$, with $\bar{u} = \hat{u}$, $\bar{v} = \hat{v}$, $2\bar{w} = \hat{w}$ respectively, as a supergroups of index 2. In these cases half-turn h around axis MN is identifying couples of points inside of simplices in such a way that we do not have a new face. It means that new groups are solid transitive although there are no simplices as their fundamental domain. Let us call their fundamental domains half-simplices and denote them by hT_4 and hT_3 (Fig.13).

The generators of the group ${}^2_2\Gamma_4(u, 2v, w)$ are

$$r_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}; r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}; h : \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \end{pmatrix}$$

and the generators of $\Gamma_{17}(2u, 4v, 2w)$ expressed by those of ${}^2_2\Gamma_4(\hat{u}, 2\hat{v}, \hat{w})$ are

$$r_0 = hr_1h, r_2 = hr_3h.$$

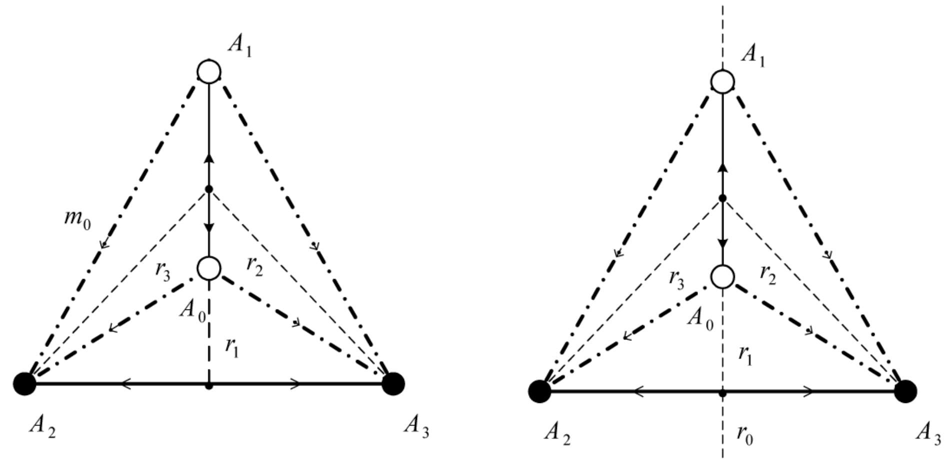

Figure 9: Simplices T_{10} and T_{17} from F4

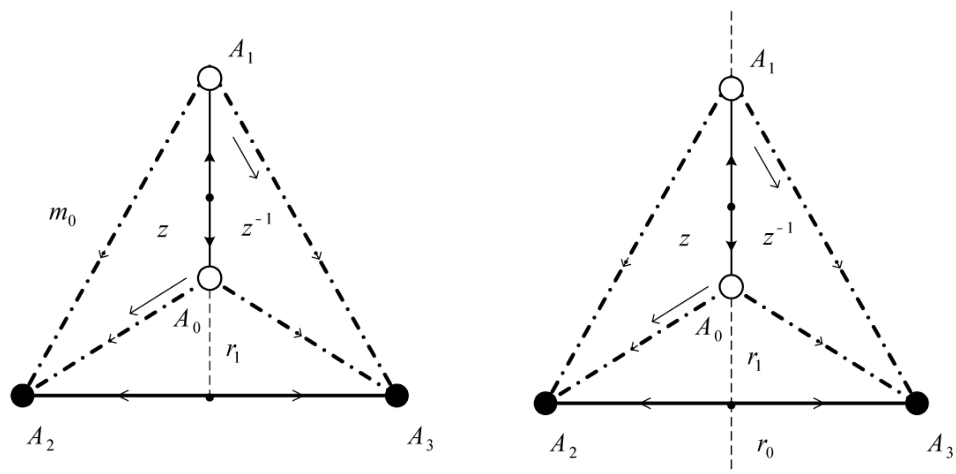

Figure 10: Simplices T_{28} and T_{38} from F4

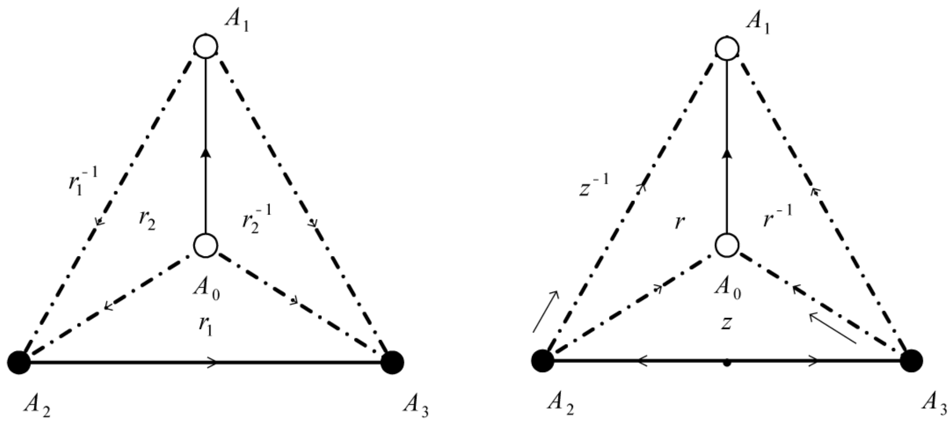

Figure 11: Simplices T_{54} and T_{57} from F4

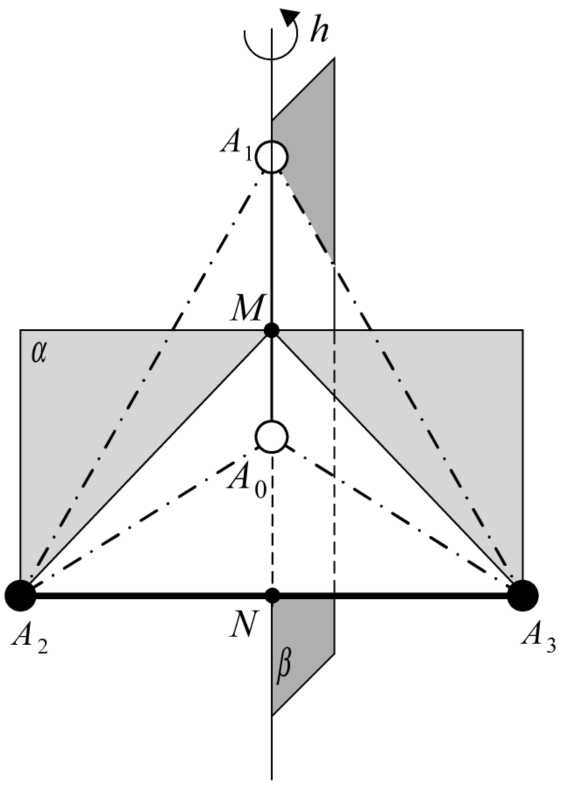

Figure 12: Halving a simplex from F4 with planes α and β

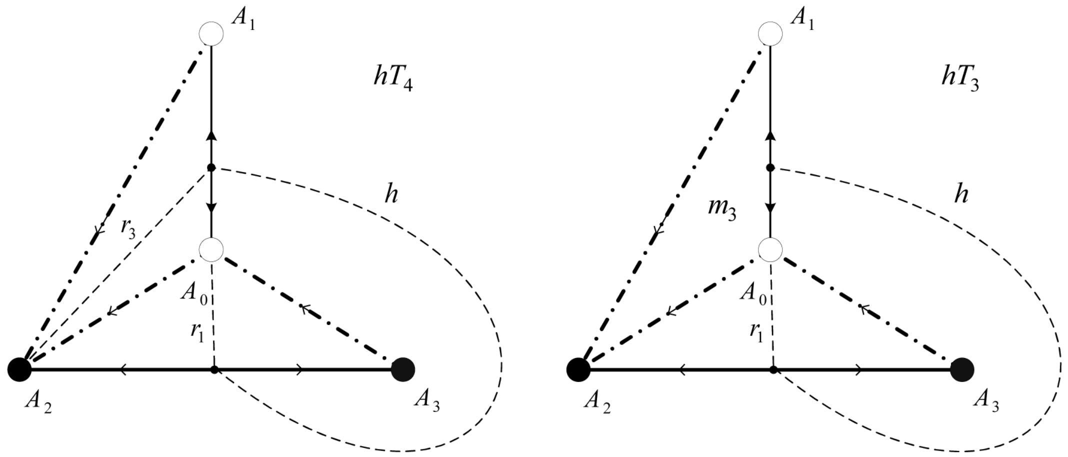

Figure 13: Half-simplices hT_4 and hT_3

For the group ${}^2_2\Gamma_3(2\hat{u}, 4\hat{v}, \hat{w})$ the generators are

$$r_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}; m_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_1 & A_2 \end{pmatrix}; h : \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \end{pmatrix},$$

while the generators of $\Gamma_{38}(2u, 4v, 2w)$ expressed by them are

$$r_0 = hr_1h, z = m_3h.$$

The only supergroup of both ${}^2_2\Gamma_4(\hat{u}, 2\hat{v}, \hat{w})$ and ${}^2_2\Gamma_3(2\hat{u}, 4\hat{v}, \hat{w})$ is the Coxeter group ${}^{mm2}_{\ \ 4}\Gamma_1(\bar{u}, 2\bar{v}, \bar{w})$ with corresponding parameters.

We can also truncate half-simplices, when rotatory parameters are such that vertices are out of the absolute. If we pair then the trunc-faces with plane reflections $\bar{m}_0$, $\bar{m}_2$ (trivial group extension) the new relations are (by the Poincaré algorithm):

- for $G_1(hO_4)$ and identified vertices A_0, A_1: $(\bar{m}_0r_1)^2 = (\bar{m}_0r_3)^2 = 1$;
  for identified vertices A_2, A_3: $(\bar{m}_2r_3)^2 = (\bar{m}_2r_1)^2 = 1$;
- for $G_1(hO_3)$ and A_0, A_1: $(\bar{m}_0r_1)^2 = (\bar{m}_0m_3)^2 = 1$;
  for A_2, A_3: $(\bar{m}_2r_1)^2 = (\bar{m}_2m_3)^2 = 1$.

Other possibility to pair the trunc-faces of half-simplices, is with half-turns $\bar{r}_0$, $\bar{r}_2$. Then we have

- for $G_2(hO_4)$ and A_0, A_1: $(r_3\bar{r}_0)^2 = (r_1\bar{r}_0)^2 = 1$;
  for A_2, A_3: $(r_1\bar{r}_2)^2 = (r_3\bar{r}_2)^2 = 1$;
- for $G_2(hO_3)$ and A_0, A_1: $(\bar{r}_0m_3)^2 = (\bar{r}_0r_1)^2 = 1$;
  for A_2, A_3: $(\bar{r}_2r_1)^2 = (\bar{r}_2m_3)^2 = 1$.

Results for Family 4 are collected in Table 2, where 1–3 are organized as before. Here, the vertex classes are denoted by $l : \{A_0, A_1\}$ and $m : \{A_2, A_3\}$. In 4–6 there are given data about possible supergroups of index 2: splitting isometry; notation of simplex supergroup; notations of trunc-simplex supergroups. If splitting of O_i gives O_j (or hO_j), then in all variants of pairings trunc-faces of O_i which are given, the trivial pairing of O_j (i.e. hO_j) provides a supergroup. Beside that, for some cases of non-trivial pairings of trunc-faces of O_i, there is non-trivial pairing of trunc-faces of O_j (i.e. hO_j) which gives supergroup. That cases are also indicated in the table.

Table 2: Family 4

1. $\mathbf{T}_{10}$:$\Gamma_{10}(2u, 8v, 4w)$ m_0: $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}$; r_1: $\begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; r_2: $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}$; r_3: $\begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $2u = \bar{u}, 4v = \bar{v}, 4w = \bar{w}$; $m_0 = m_\alpha m_1 m_\alpha, r_1 = m_1 m_\beta, r_2 = m_\beta(m_3 m_\alpha)m_\beta, r_3 = m_3 m_\alpha$
3. l) **I**: $\bar{m}_0$, $\bar{m}_1$ m) **I**: $\bar{m}_2$, $\bar{m}_3$; **II**: s: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^0_3 & T^1_3 & T^2_3 \end{pmatrix}$
5. m_β; Γ_2; $G_1(O_2)$
1. $\mathbf{T}_{17}$:$\Gamma_{17}(2u, 4v, 2w)$ r_0: $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_3 & A_2 \end{pmatrix}$; r_1: $\begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; r_2: $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}$; r_3: $\begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $2u = \bar{u}, 2v = \bar{v}, 2w = \bar{w}$; $m_0 = m_\alpha(m_\beta m_1)m_\alpha, r_1 = m_\beta m_1, r_2 = m_\beta(m_3 m_\alpha)m_\beta, r_3 = m_\alpha m_3$
3. l) **I**: $\bar{m}_0$, $\bar{m}_1$; **II**: s_1: $\begin{pmatrix} T^1_0 & T^2_0 & T^3_0 \\ T^0_1 & T^2_1 & T^3_1 \end{pmatrix}$; **III**: g_1: $\begin{pmatrix} T^1_0 & T^2_0 & T^3_0 \\ T^0_1 & T^3_1 & T^2_1 \end{pmatrix}$ m) **I**: $\bar{m}_2$, $\bar{m}_3$; **II**: s_2: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^0_3 & T^1_3 & T^2_3 \end{pmatrix}$; **III**: g_2: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^1_3 & T^0_3 & T^2_3 \end{pmatrix}$
4. h_α; Γ_{35}; $G_1(O_{35})$, $G_2(O^m_{17}) \to G_2(O^l_{35})$
5. h_β, Γ_{35}; $G_1(O_{35})$, $G_2(O^l_{17}) \to G_2(O^l_{35})$
6. h, ${}^2_2\Gamma_4$; $G_1(hO_4)$, $G_j(O^t_{17}) \to G_2(hO^t_4)$, $j = 2, 3$, $t \in \{l, m\}$
1. $\mathbf{T}_{28}$:$\Gamma_{28}(2u, 8v, 4w)$ m_0 : $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}$; r_1 : $\begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; z : $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $2u = \bar{u}, 4v = \bar{v}, 4w = \bar{w}$; $m_0 = m_\alpha m_1 m_\alpha, r_1 = m_1 m_\beta, z = m_3 m_\alpha m_\beta$
3. l) **I**: $\bar{m}_0$, $\bar{m}_1$ m) **I**: $\bar{m}_2$, $\bar{m}_3$; **II**: s: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^0_3 & T^1_3 & T^2_3 \end{pmatrix}$
5. m_β; Γ_2; $G_1(O_2)$
1. $\mathbf{T}_{38}$:$\Gamma_{38}(2u, 4v, 2w)$ r_0 : $\begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_3 & A_2 \end{pmatrix}$; r_1 : $\begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; z : $\begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_2 \end{pmatrix}$
2. $2u = \bar{u}, 2v = \bar{v}, 2w = \bar{w}$; $r_0 = m_\alpha m_\beta m_1 m_\alpha, r_1 = m_\beta m_1, z = m_\beta m_\alpha m_3$
3. l) **I**: $\bar{m}_0$, $\bar{m}_1$; **II**: $\bar{z}$: $\begin{pmatrix} T^1_0 & T^2_0 & T^3_0 \\ T^0_1 & T^3_1 & T^2_1 \end{pmatrix}$; **III**: s_1: $\begin{pmatrix} T^1_0 & T^2_0 & T^3_0 \\ T^0_1 & T^2_1 & T^3_1 \end{pmatrix}$ m) **I**: $\bar{m}_2$, $\bar{m}_3$; **II**: $\bar{z}_1$: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^1_3 & T^0_3 & T^2_3 \end{pmatrix}$; **III**: h_2: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^1_2 & T^0_2 & T^3_2 \end{pmatrix}$, h_3: $\begin{pmatrix} T^0_3 & T^1_3 & T^2_3 \\ T^1_3 & T^0_3 & T^2_3 \end{pmatrix}$; **IV**: s: $\begin{pmatrix} T^0_2 & T^1_2 & T^3_2 \\ T^0_3 & T^1_3 & T^2_3 \end{pmatrix}$
4. h_α; Γ_6; $G_1(O_6)$, $G_4(O^m_{38}) \to G_2(O^l_6)$
6. h, ${}^2_2\Gamma_3$; $G_1(hO_3)$, $G_3(O^l_{38}) \to G_2(hO^l_3)$, $G_j(O^m_{38}) \to G_2(hO^m_3)$, $j = 2, 3, 4$

1. $\mathbf{T}_{54}$:$\Gamma_{54}(u,4v,w)$ $r_1 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}; r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_2 \end{pmatrix}$
2. $u = \bar{u}, 2v = \bar{v}, w = \bar{w};$ $r_1 = m_\alpha m_1, r_2 = m_\beta m_3$
3. $l)$ **I**: $\bar{m}_0, \bar{m}_1$; **II**: s_1: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^2 & T_1^3 \end{pmatrix}$; **III**: h_0: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_0^1 & T_0^3 & T_0^2 \end{pmatrix}$, h_1: $\begin{pmatrix} T_1^0 & T_1^2 & T_1^3 \\ T_1^0 & T_1^3 & T_1^2 \end{pmatrix}$; **IV**: z_1: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^3 & T_1^2 \end{pmatrix}$ $m)$ **I**: $\bar{m}_2, \bar{m}_3$; **II**: s_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_3^0 & T_3^1 & T_3^2 \end{pmatrix}$; **III**: h_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_2^1 & T_2^0 & T_2^3 \end{pmatrix}$, h_3: $\begin{pmatrix} T_3^0 & T_3^1 & T_3^2 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$; **IV**: z_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$
4. m_α; Γ_{20}; $G_1(O_{20})$, $G_j(O_{54}^m) \to G_2(O_{20}^l)$, $j = 2,4$
5. m_β; Γ_{20}; $G_1(O_{20})$, $G_j(O_{54}^l) \to G_2(O_{20}^l)$, $j = 2,4$

1. $\mathbf{T}_{57}$:$\Gamma_{57}(u,4v,2w)$ $z : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}; r : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_2 \end{pmatrix}$
2. $u = \bar{u}, 2v = \bar{v}, 2w = \bar{w};$ $z = m_\beta m_\alpha m_1, r = m_\beta m_3$
3. $l)$ **I**: $\bar{m}_0, \bar{m}_1$; **II**: $\bar{z}_1$: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^3 & T_1^2 \end{pmatrix}$; **III**: h_0: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_0^1 & T_0^3 & T_0^2 \end{pmatrix}$, h_1: $\begin{pmatrix} T_1^0 & T_1^2 & T_1^3 \\ T_1^0 & T_1^3 & T_1^2 \end{pmatrix}$; **IV**: s_1: $\begin{pmatrix} T_0^1 & T_0^2 & T_0^3 \\ T_1^0 & T_1^2 & T_1^3 \end{pmatrix}$ $m)$ **I**: $\bar{m}_2, \bar{m}_3$; **II**: s_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_3^0 & T_3^1 & T_3^2 \end{pmatrix}$; **III**: h_2: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_2^1 & T_2^0 & T_2^3 \end{pmatrix}$, h_3: $\begin{pmatrix} T_3^0 & T_3^1 & T_3^2 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$; **IV**: $\bar{z}_2$: $\begin{pmatrix} T_2^0 & T_2^1 & T_2^3 \\ T_3^1 & T_3^0 & T_3^2 \end{pmatrix}$
4. h_α; Γ_{24}; $G_1(O_{24})$, $G_2(O_{57}^m) \to G_2(O_{35}^l)$

Note that symmetries of a simplex induce symmetries of the fundamental domain of its vertex figure. So, it is obvious that if the group of a simplex is the maximal one, then there are no symmetries of the fundamental domain of the vertex figure according to the face pairing. Consequently, only trivial pairing of trunc-faces is possible. Also symmetries of T_{17} and T_{38} leading to hT_4 and hT_3 show new, unexpected symmetry for each of the vertex figures of T_{17} and T_{38}, respectively. Namely, (in all cases) this is rotation in the polar plane of the vertex, as it is indicated in Figures 14 and 15. It means that there is one more possibility for pairing trunc-faces of O_{17} and O_{38} with face pairing identifications, not mentioned yet in [14].

In the case of trunc-simplex O_{17} (Fig. 16) the identifying isometries are glide reflections g_1, g_2, and the new relations for A_0 (variant **III** in Table 2) are

$$g_1 r_3 g_1 r_2 = g_1 r_0 g_1^{-1} r_1 = 1,$$

while for A_2 (variant **III**) these are

$$g_2 r_1 g_2 r_0 = g_2 r_2 g_2^{-1} r_3 = 1.$$

Here g_1, g_2 are expressed by generators of hO_4^l

$$g_1 = h\bar{m}_0, g_2 = h\bar{m}_2.$$

For trunc-simplex O_{38} screw-motion s_1 identify the trunc-faces of vertices A_0 and A_1 (variant **III**), so the relations are

$$s_1 z s_1 z^{-1} = s_1 r_0 s_1^{-1} r_1 = 1$$

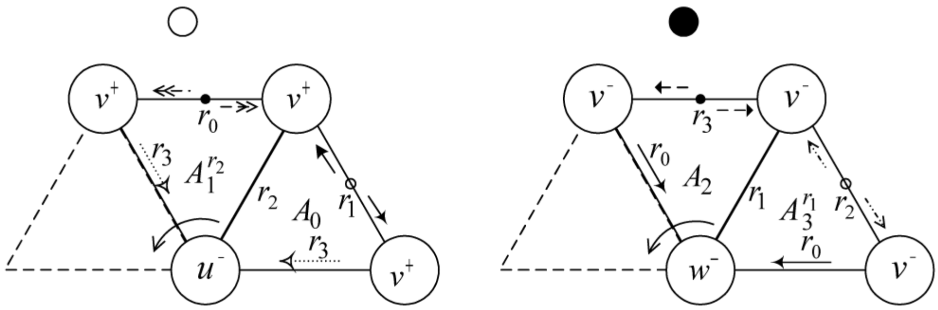

Figure 14: Vertex figures of simplex T_{17}

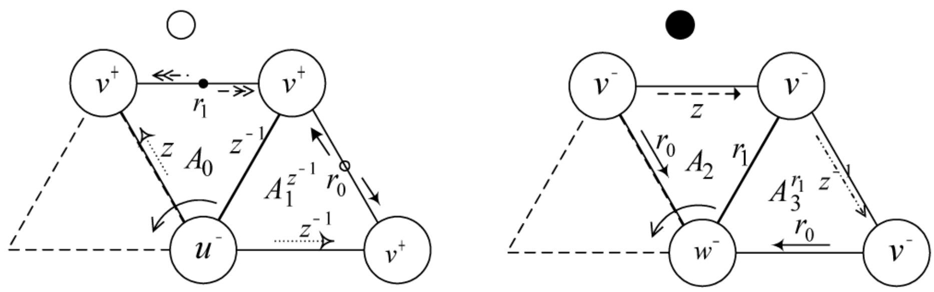

Figure 15: Vertex figures of simplex T_{38}

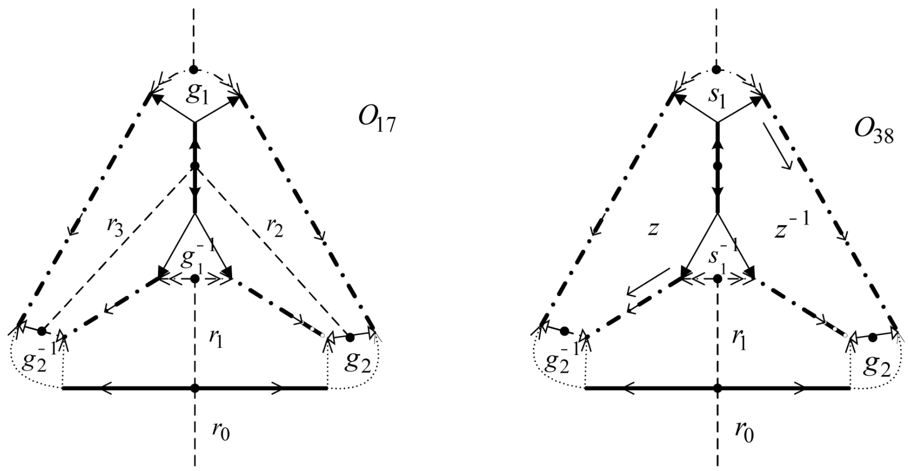

Figure 16: Trunc-simplices O_{17} and O_{38}

and s_1 is expressed by generators of hO_3^{II}

$$s_1 = h\bar{r}_0.$$

Identification of vertices A_2 and A_3 in this way is equivalent to one of the considered cases in [14] (here denoted by **II**).

4. Realization of family F6 on the base of Coxeter groups

4.1 If the 4-dimensional real vector space is denoted by V^4 and its dual space of its linear forms by $\mathcal{V}_4$, then the projective 3-space $P^3(V^4, \mathcal{V}_4)$ can be introduced in the usual way. The 1-dimensional subspaces of V^4 (or the 3-subspaces of $\mathcal{V}_4$) represent the points of P^3, and the 1-subspaces of $\mathcal{V}_4$ (or the 3-subspaces of V^4) represent the planes of P^3. The point $X(\mathbf{x})$ and the plane $\alpha(a)$ are incident iff $\mathbf{x}a = 0$, i.e. the value of the linear form a on the vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in V^4 \setminus \{0\}, a \in \mathcal{V}_4 \setminus \{0\}$). The straight lines of P^3 are characterized by 2-subspaces of V^4 or of $\mathcal{V}_4$, respectively. If $\{\mathbf{e_i}\}$ is a basis on V^4 and $\left\{e^j\right\}$ is its dual basis on $\mathcal{V}_4$, i.e. $\mathbf{e_i}e^j = \delta_i^j$ (the Kronecker symbol), then the form $a = e^j a_j$ takes the value $\mathbf{x}a = x^i a_i$ on the vector $\mathbf{x} = x^i\mathbf{e_i}$. We use the summation convention for the same upper and lower indices.

We can introduce projective metric in P^3 by giving a bilinear form

$$\langle ; \rangle : \mathcal{V}_4 \times \mathcal{V}_4 \to R, \ \left\langle b^i u_i ; b^j v_j \right\rangle = u_i b^{ij} v_j$$

where $\left(\left\langle b^i ; b^j \right\rangle\right) = \left(b^{ij}\right)$ is the Schläfli matrix, and the basis $\left\{b^i\right\}$ in $\mathcal{V}_4$ represents planes containing simplex faces opposite to the vertices A_i, respectively. Vectors $\mathbf{a_j}$ of the dual basis $\left\{\mathbf{a_j}\right\}$ in V^4, defined by $\mathbf{a_j} b^i = \delta_j^i$, represent the vertices A_j of the simplex. The induced bilinear form

$$\langle ; \rangle : V^4 \times V^4 \to R, \ \left\langle x^i \mathbf{a_i} ; y^j \mathbf{a_j} \right\rangle = x^i a_{ij} y^j$$

is defined by the matrix $\left(\left\langle \mathbf{a_i} ; \mathbf{a_j} \right\rangle\right) = \left(a_{ij}\right)$ the inverse of $\left(b^{ij}\right)$.

We assume, that the bilinear form $\langle ; \rangle$ is of signature $(+, +, +, -)$ which characterizes the hyperbolic metric, or of $(+, +, +, +)$ for elliptic (spherical) metric, or of $(+, +, +, 0)$ for Euclidean geometry (see [9] for the other cases).

It is well-known that the bilinear form induces the distance and the angle measure of the 3-space. Let $X(\mathbf{x})$ and $Y(\mathbf{y})$ be two points in the projective space P^3. Then their distance $d(\mathbf{x}, \mathbf{y})$ is determined by

$$\cos\left(d(\mathbf{x}, \mathbf{y})\right) = \frac{\langle \mathbf{x} ; \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x} ; \mathbf{x} \rangle \langle \mathbf{y} ; \mathbf{y} \rangle}} \ \text{ and } \ \mathrm{ch}\left(d(\mathbf{x}, \mathbf{y})\right) = -\frac{\langle \mathbf{x} ; \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x} ; \mathbf{x} \rangle \langle \mathbf{y} ; \mathbf{y} \rangle}}$$

for the elliptic and hyperbolic case, respectively.

For the four simplices in family F6 the Schläfli matrix is

$$b^{ij} = \left\langle b^i ; b^j \right\rangle = \begin{pmatrix} 1 & -\cos\frac{2\pi}{\bar{x}} & -\cos\frac{\pi}{\bar{w}} & -\cos\frac{\pi}{\bar{v}} \\ -\cos\frac{2\pi}{\bar{x}} & 1 & -\cos\frac{\pi}{\bar{w}} & -\cos\frac{\pi}{\bar{v}} \\ -\cos\frac{\pi}{\bar{w}} & -\cos\frac{\pi}{\bar{w}} & 1 & -\cos\frac{\pi}{\bar{u}} \\ -\cos\frac{\pi}{\bar{v}} & -\cos\frac{\pi}{\bar{v}} & -\cos\frac{\pi}{\bar{u}} & 1 \end{pmatrix}.$$

But, because of symmetries of these simplices, the Schläfli matrix for the half of them, the simplex T_{F6} (Fig. 1) can serve for investigating the space of realization. That matrix is

$$\tilde{b}^{ij} = \begin{pmatrix} 1 & -\cos\frac{\pi}{\bar{x}} & 0 & 0 \\ -\cos\frac{\pi}{\bar{x}} & 1 & -\cos\frac{\pi}{\bar{w}} & -\cos\frac{\pi}{\bar{v}} \\ 0 & -\cos\frac{\pi}{\bar{w}} & 1 & -\cos\frac{\pi}{\bar{u}} \\ 0 & -\cos\frac{\pi}{\bar{v}} & -\cos\frac{\pi}{\bar{u}} & 1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -\cos\frac{\pi}{\bar{x}} \\ 0 & 1 & -\cos\frac{\pi}{\bar{u}} & -\cos\frac{\pi}{\bar{w}} \\ 0 & -\cos\frac{\pi}{\bar{u}} & 1 & -\cos\frac{\pi}{\bar{v}} \\ -\cos\frac{\pi}{\bar{x}} & -\cos\frac{\pi}{\bar{w}} & -\cos\frac{\pi}{\bar{v}} & 1 \end{pmatrix},$$

the latter is by reordering the basis. It is obvious that it has positive definite 3×3 minor

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\cos\frac{\pi}{\bar{u}} \\ 0 & -\cos\frac{\pi}{\bar{u}} & 1 \end{vmatrix}$$

so, the signature depends of the sign of the determinant. Its value is

$$\det \tilde{b}^{ij} = \left\{1 - \cos^2\frac{\pi}{\bar{u}} - \cos^2\frac{\pi}{\bar{v}} - \cos^2\frac{\pi}{\bar{w}} - 2\cos\frac{\pi}{\bar{u}}\cos\frac{\pi}{\bar{v}}\cos\frac{\pi}{\bar{w}}\right\} +$$

$$\cos\frac{\pi}{\bar{x}}\left[\cos\frac{\pi}{\bar{x}}\left(1 - \cos^2\frac{\pi}{\bar{u}}\right)\right] =$$

$$= \left\{1 - \cos^2\frac{\pi}{\bar{v}} - \cos^2\frac{\pi}{\bar{x}}\right\} - \cos^2\frac{\pi}{\bar{u}}\left(1 - \cos^2\frac{\pi}{\bar{x}} - \cos^2\frac{\pi}{\bar{w}} - 2\cos\frac{\pi}{\bar{u}}\cos\frac{\pi}{\bar{v}}\cos\frac{\pi}{\bar{w}}\right) =$$

$$= \left\{1 - \cos^2\frac{\pi}{\bar{w}} - \cos^2\frac{\pi}{\bar{x}}\right\} - \cos^2\frac{\pi}{\bar{u}}\left(1 - \cos^2\frac{\pi}{\bar{x}} - \cos^2\frac{\pi}{\bar{v}} - 2\cos\frac{\pi}{\bar{u}}\cos\frac{\pi}{\bar{v}}\cos\frac{\pi}{\bar{w}}\right),$$

where the terms in brackets { } are equal to the minor determinants B_0, B_2 and B_3 of vertex figures for vertices $\bar{A}_0$, $\bar{A}_2$ and $\bar{A}_3$, respectively. The above expression show that, if a vertex figure is of hyperbolic signature $(+,+,-)$, i.e. the vertex is outer, then the determinant $\det \tilde{b}^{ij}$ is negative. It means, then the signature of b^{ij} is $(+,+,+,-)$, so the geometry is already hyperbolic.

In considering sign of B_0, B_2 and B_3 it will be used the following identity

$$D = 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma =$$

$$= 1 - \frac{\cos 2\alpha + 1}{2} - \frac{\cos 2\beta + 1}{2} - \cos\gamma\,(2\cos\alpha\cos\beta + \cos\gamma) =$$

$$= -\cos(\alpha+\beta)\cos(\alpha-\beta) - \cos\gamma\,[\cos(\alpha+\beta) + \cos(\alpha-\beta) + \cos\gamma] =$$

$$= -\,[\cos(\alpha+\beta) + \cos\gamma]\,[\cos(\alpha-\beta) + \cos\gamma] =$$

$$= -4\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{\alpha+\beta-\gamma}{2}\cos\frac{\alpha-\beta+\gamma}{2}\cos\frac{-\alpha+\beta+\gamma}{2}.$$

We are working with angles in the interval $\left[0, \frac{\pi}{2}\right]$, so $\cos\frac{\alpha+\beta-\gamma}{2} > 0$, $\cos\frac{\alpha-\beta+\gamma}{2} > 0$, $\cos\frac{-\alpha+\beta+\gamma}{2} > 0$ and $-\frac{\sqrt{2}}{2} \le \cos\frac{\alpha+\beta+\gamma}{2} \le 1$. It follows that $D > 0$ for $\alpha+\beta+\gamma > \pi$, $D = 0$ for $\alpha+\beta+\gamma = \pi$, $D < 0$ for $\alpha+\beta+\gamma < \pi$. If D is determinant of vertex figure it describes a vertex which is proper, ideal, outer, respectively.

Condition for $\bar{A}_0$ is outer vertex, if above we take $\alpha = \frac{\pi}{\bar{u}}$, $\beta = \frac{\pi}{\bar{v}}$, $\gamma = \frac{\pi}{\bar{w}}$ with angel sum smaller then π. Similar condition stands for $\bar{A}_2$ and $\bar{A}_3$ when $\alpha = \frac{\pi}{\bar{x}}$, $\beta = \frac{\pi}{\bar{v}}$, $\gamma = \frac{\pi}{2}$, and $\alpha = \frac{\pi}{\bar{x}}$, $\beta = \frac{\pi}{\bar{w}}$, $\gamma = \frac{\pi}{2}$, respectively.

4.2 Similarly as in [10] we can express the generators of the Coxeter group

$${}^{m}_{2}\Gamma_1(2\bar{u}, 2\bar{v}, 2\bar{w}, 2\bar{v}) = (m'_0, m'_1, m'_2, m'_3 - {m'}^2_0 = {m'}^2_1 = {m'}^2_2 = {m'}^2_3 =$$

$$= (m'_0 m'_1)^{\bar{x}} = (m'_0 m'_2)^2 = (m'_0 m'_3)^2 = (m'_1 m'_2)^{\bar{w}} =$$

$$= (m'_1 m'_3)^{\bar{v}} = (m'_2 m'_3)^{\bar{u}} = 1, \bar{u} \ge 2, \bar{w} \ge \bar{v} \ge 2, \bar{x} \ge 3)$$

by matrices in basis $\bar{A}_i(\mathbf{a}_i)$ $(i = 0, 1, 2, 3)$. Matrices for plane reflections m'_0, m'_1, m'_2, m'_3 are denoted by M'_0, M'_1, N'_2, P'_3, respectively:

$$M'_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M'_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_1^0 & -1 & m_1^2 & m_1^3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$N'_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ n_2^0 & n_2^1 & -1 & n_2^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P'_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_3^0 & p_3^1 & p_3^2 & -1 \end{pmatrix}.$$

Here, $m_1^0 = 2\cos\frac{2\pi}{\bar{x}}$, $m_1^2 = 2\cos\frac{\pi}{\bar{w}}$, $m_1^3 = 2\cos\frac{\pi}{\bar{v}}$, $n_2^0 = n_2^1 = 2\cos\frac{\pi}{\bar{w}}$, $n_2^3 = 2\cos\frac{\pi}{\bar{u}}$, $p_3^0 = p_3^1 = 2\cos\frac{\pi}{\bar{v}}$, $p_3^2 = 2\cos\frac{\pi}{\bar{u}}$. Namely, we have applied the above relations to the corresponding matrices, e.g.

$$\left(M'_0N'_2\right)^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ n_2^0 & n_2^1 & -1 & n_2^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}^2 =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ n_2^1 - n_2^0 & n_2^0 - n_2^1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If the polar plane of the vertex $\bar{A}_3(\mathbf{a}_3)$ exists (i.e. $\bar{A}_3$ is hyperbolic), its form will be denoted by $\bar{a}_3$. It holds for the scalar products of forms $\langle \bar{a}_3, b^i \rangle = 0$ for $i = 0, 1, 2$, $\langle \bar{a}_3, b^3 \rangle \neq 0$; $\bar{a}_3 = b^0a_{03} + b^1a_{13} + b^2a_{23} + b^3a_{33} =: b^ia_{i3}$ in the basis $\{b^i\}$ of forms in $\mathcal{V}_4$ (such that $\mathbf{a}_\mathbf{i}b^j = \delta_i^j$), and $b_*^i = b^{ij}\mathbf{a}_\mathbf{j}$ express the plane - point polarity by $b^{ij} = \langle b^i, b^j \rangle$. Its inverse $\left(b^{ij}\right)^{-1} = \left(a_{ij}\right)$, if exists, expresses the above equation $\bar{a}_i = b^ja_{ij}$ for $i = 3$, as example.

a) The reflection $\bar{m}_3$ in the plane $\bar{a}_3$ can be expressed in the form basis $\{b^i\}$, first by a matrix $\bar{\mathbf{M}}_3 \to \left(e_j^i\right)$ as follows $\bar{\mathbf{M}}_3 : b^i \to b^je_j^i$

$$\bar{\mathbf{M}}_3 : \begin{pmatrix} b^0 & b^1 & b^2 & b^3 \end{pmatrix} \longrightarrow \begin{pmatrix} b^0 & b^1 & b^2 & b^3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & e_0^3 \\ 0 & 1 & 0 & e_1^3 \\ 0 & 0 & 1 & e_2^3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and by its inverse matrix $\left(e_j^i\right)^{-1}$, in the vector basis $\{\mathbf{a}_j\}$. It is the same, since $\bar{m}_3$ is an involution (as a reflection in a plane)

$$\bar{\mathbf{M}}_3 : \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & e_0^3 \\ 0 & 1 & 0 & e_1^3 \\ 0 & 0 & 1 & e_2^3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}.$$

Since $\bar{m}_3$ commutes with m'_0, m'_1 and m'_2, e.g. from $\bar{m}_3m'_1 = m'_1\bar{m}_3$, (i.e. $\bar{\mathbf{M}}_3M'_1 = M'_1\bar{\mathbf{M}}_3$) yields

$$\begin{pmatrix} 1 & 0 & 0 & e_0^3 \\ m_1^0 & -1 & m_1^2 & m_1^3 + e_1^3 \\ 0 & 0 & 1 & e_2^3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & e_0^3 \\ m_1^0 & -1 & m_1^2 & m_1^0e_0^3 - e_1^3 + m_1^2e_2^3 - m_1^3 \\ 0 & 0 & 1 & e_2^3 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

i.e. $2\left(m_1^3 + e_1^3\right) = m_1^0 e_0^3 + m_1^2 e_2^3$. Similarly, we get from $\bar{\mathbf{M}}_3 N_2' = N_2' \bar{\mathbf{M}}_3$, $2\left(n_2^3 + e_2^3\right) = n_2^0 e_0^3 + n_2^1 e_1^3$. From $\bar{\mathbf{M}}_3 M_0' = M_0' \bar{\mathbf{M}}_3$ we obtain

$$\begin{pmatrix} 0 & 1 & 0 & e_0^3 \\ 1 & 0 & 0 & e_1^3 \\ 0 & 0 & 1 & e_2^3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & e_1^3 \\ 1 & 0 & 0 & e_0^3 \\ 0 & 0 & 1 & e_2^3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e. $e_0^3 = e_1^3$. Solving the system of the unknown parameters e_0^3, e_1^3 and e_2^3, the determinant of the system is

$$D_3 = \left(n_2^1 m_1^2 - 4\right) + \left(n_2^0 m_1^2 + 2m_1^0\right) = 2\cos\frac{\pi}{\bar{w}}\left(4\cos\frac{\pi}{\bar{w}}\right) + 2\left(2\cos\frac{2\pi}{\bar{x}} - 2\right) =$$

$$= 8\cos^2\frac{\pi}{\bar{w}} + 8\cos^2\frac{\pi}{\bar{x}} - 8 = 8\left(\cos\left(\frac{\pi}{\bar{w}} + \frac{\pi}{\bar{x}}\right)\cos\left(\frac{\pi}{\bar{w}} - \frac{\pi}{\bar{x}}\right)\right) > 0.$$

Thus, we can express

$$e_0^3 = e_1^3 = \frac{1}{D_3}\begin{vmatrix} 0 & -1 & 0 \\ 2n_2^3 & n_2^1 & -2 \\ 2m_1^3 & -2 & m_1^2 \end{vmatrix} = \frac{\cos\frac{\pi}{\bar{u}}\cos\frac{\pi}{\bar{w}} + \cos\frac{\pi}{\bar{v}}}{\cos\left(\frac{\pi}{\bar{w}} + \frac{\pi}{\bar{x}}\right)\cos\left(\frac{\pi}{\bar{w}} - \frac{\pi}{\bar{x}}\right)},$$

$$e_2^3 = \frac{1}{D_3}\begin{vmatrix} 1 & -1 & 0 \\ n_2^0 & n_2^1 & 2n_2^3 \\ m_1^0 & -2 & m_1^3 \end{vmatrix} = \frac{\cos\frac{\pi}{\bar{w}}\cos\frac{\pi}{\bar{v}} + \cos\frac{\pi}{\bar{u}}}{\cos\left(\frac{\pi}{\bar{w}} + \frac{\pi}{\bar{x}}\right)\cos\left(\frac{\pi}{\bar{w}} - \frac{\pi}{\bar{x}}\right)}.$$

b) The reflection $\bar{m}_2$ in the plane $\bar{a}_2$ can be described in the form basis $\left\{b^i\right\}$, by a matrix $\bar{\mathbf{M}}_2 \to \left(d_j^i\right)$, $\bar{\mathbf{M}}_2 : b^i \to b^j d_j^i$,

$$\bar{\mathbf{M}}_2 : \begin{pmatrix} b^0 & b^1 & b^2 & b^3 \end{pmatrix} \longrightarrow \begin{pmatrix} b^0 & b^1 & b^2 & b^3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & d_0^2 & 0 \\ 0 & 1 & d_1^2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & d_3^2 & 1 \end{pmatrix},$$

and in the vector basis $\left\{\mathbf{a}_j\right\}$ by

$$\bar{\mathbf{M}}_2 : \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & d_0^2 & 0 \\ 0 & 1 & d_1^2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & d_3^2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}.$$

Again, $\bar{m}_2$ commutes with m_0', m_1' and m_3', these determine d_0^2, d_1^2, d_3^2. Similarly as before $d_0^2 = d_1^2$, $2\left(m_1^2 + d_1^2\right) = m_1^0 d_0^2 + m_1^3 d_3^2$, $2\left(d_3^2 + p_3^2\right) = p_3^0 d_0^2 + p_3^1 d_1^2$. Here, determinant of system is

$$D_2 = 4 - m_1^3 p_3^1 - 2m_1^0 - m_1^3 p_3^0 = -8\left(\cos\left(\frac{\pi}{\bar{x}} + \frac{\pi}{\bar{v}}\right)\cos\left(\frac{\pi}{\bar{x}} - \frac{\pi}{\bar{v}}\right)\right)$$

and we get

$$d_0^2 = d_1^2 = \frac{1}{D_2}\begin{vmatrix} 0 & -1 & 0 \\ 2m_1^2 & -2 & m_1^3 \\ 2p_3^2 & p_3^1 & -2 \end{vmatrix} = \frac{\cos\frac{\pi}{\bar{w}} + \cos\frac{\pi}{\bar{v}}\cos\frac{\pi}{\bar{u}}}{\cos\left(\frac{\pi}{\bar{x}} + \frac{\pi}{\bar{v}}\right)\cos\left(\frac{\pi}{\bar{x}} - \frac{\pi}{\bar{v}}\right)},$$

$$d_3^2 = \frac{1}{D_2}\begin{vmatrix} 1 & -1 & 0 \\ m_1^0 & -2 & 2m_1^2 \\ p_3^0 & p_3^1 & 2p_3^2 \end{vmatrix} = \frac{2\left(\cos\frac{\pi}{\bar{u}} + \cos\frac{\pi}{\bar{v}}\cos\frac{\pi}{\bar{w}} - \cos^2\frac{\pi}{\bar{x}}\cos\frac{\pi}{\bar{u}}\right)}{\cos\left(\frac{\pi}{\bar{x}} + \frac{\pi}{\bar{v}}\right)\cos\left(\frac{\pi}{\bar{x}} - \frac{\pi}{\bar{v}}\right)}.$$

c) Finally, the reflection $\bar{m}_0$ in the plane $\bar{a}_0$ can be expressed by $\bar{\mathbf{M}}_0 : b^i \to b^j c_j^i$,

$$\bar{\mathbf{M}}_0 : \begin{pmatrix} b^0 & b^1 & b^2 & b^3 \end{pmatrix} \longrightarrow \begin{pmatrix} b^0 & b^1 & b^2 & b^3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ c_1^0 & 1 & 0 & 0 \\ c_2^0 & 0 & 1 & 0 \\ c_3^0 & 0 & 0 & 1 \end{pmatrix},$$

and by

$$\bar{\mathbf{M}}_0 : \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ c_1^0 & 1 & 0 & 0 \\ c_2^0 & 0 & 1 & 0 \\ c_3^0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix}.$$

Reflection $\bar{m}_0$ commutes with m_1', m_2' and m_3'. So, $2\left(c_1^0 + m_1^0\right) = m_1^2 c_2^0 + m_1^3 c_3^0$, $2\left(c_2^0 + n_2^0\right) = n_2^1 c_1^0 + n_2^3 c_3^0$, $2\left(c_3^0 + p_3^0\right) = p_3^1 c_1^0 + p_3^2 c_2^0$. Determinant D_0 is equal to

$$D_0 = -8 + m_1^2 n_2^3 p_3^1 + m_1^3 n_2^1 p_3^2 + 2n_2^3 p_3^2 + 2m_1^2 n_2^1 + 2m_1^3 p_3^1 = 32 \cos \frac{1}{2}\left(\frac{\pi}{\bar{u}} + \frac{\pi}{\bar{v}} + \frac{\pi}{\bar{w}}\right) \cdot$$

$$\cdot \cos \frac{1}{2}\left(\frac{\pi}{\bar{u}} + \frac{\pi}{\bar{v}} - \frac{\pi}{\bar{w}}\right) \cos \frac{1}{2}\left(\frac{\pi}{\bar{u}} - \frac{\pi}{\bar{v}} + \frac{\pi}{\bar{w}}\right) \cos \frac{1}{2}\left(-\frac{\pi}{\bar{u}} + \frac{\pi}{\bar{v}} + \frac{\pi}{\bar{w}}\right).$$

Hence,

$$c_1^0 = \frac{1}{D_0} \begin{vmatrix} 2m_1^0 & m_1^2 & m_1^3 \\ 2n_2^0 & -2 & n_2^3 \\ 2p_3^0 & p_3^2 & -2 \end{vmatrix} = \frac{1}{D_0} \left[32 \cos^2 \frac{\pi}{\bar{x}} \left(1 - \cos^2 \frac{\pi}{\bar{u}}\right) - \right.$$

$$\left. -16 \left(1 - \cos^2 \frac{\pi}{\bar{u}} - \cos^2 \frac{\pi}{\bar{v}} - \cos^2 \frac{\pi}{\bar{w}} - 2 \cos \frac{\pi}{\bar{u}} \cos \frac{\pi}{\bar{v}} \cos \frac{\pi}{\bar{w}}\right) \right],$$

$$c_2^0 = \frac{1}{D_0} \begin{vmatrix} -2 & 2m_1^0 & m_1^3 \\ n_2^1 & 2n_2^0 & n_2^3 \\ p_3^1 & 2p_3^0 & -2 \end{vmatrix} = \frac{32 \cos^2 \frac{\pi}{\bar{x}} \left(\cos \frac{\pi}{\bar{u}} \cos \frac{\pi}{\bar{v}} + \cos \frac{\pi}{\bar{w}}\right)}{D_0},$$

$$c_3^0 = \frac{1}{D_0} \begin{vmatrix} -2 & m_1^2 & 2m_1^0 \\ n_2^1 & -2 & 2n_2^0 \\ p_3^1 & p_3^2 & 2p_3^0 \end{vmatrix} = \frac{32 \cos^2 \frac{\pi}{\bar{x}} \left(\cos \frac{\pi}{\bar{v}} + \cos \frac{\pi}{\bar{u}} \cos \frac{\pi}{\bar{w}}\right)}{D_0}.$$

Summary. The matrices above, expressed by the D-matrix $(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$ with $2 \leq \bar{u}$, $2 \leq \bar{v} < \bar{w}$, $3 \leq \bar{x}$, i.e. by angles $\frac{\pi}{\bar{u}}, \frac{\pi}{\bar{v}}, \frac{\pi}{\bar{w}}, \frac{\pi}{\bar{x}}$, describes the extended reflection groups ${}_2^m\bar{\Gamma}_1\,(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$, which exist if conditions $\frac{1}{\bar{u}} + \frac{1}{\bar{v}} + \frac{1}{\bar{w}} < 1$, $\frac{1}{\bar{x}} + \frac{1}{\bar{v}} < \frac{1}{2}$, $\frac{1}{\bar{x}} + \frac{1}{\bar{w}} < \frac{1}{2}$, for vertices to be hyperbolic, are fulfilled.

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References

[1] B. Maskit, On Poincarés theorem for fundamental polygons, Adv. in Math. 7 (1971) 219–230.

[2] E. Molnár, Eine Klasse von hyperbolischen Raumgruppen, Beiträge zur Algebra und Geometrie 30 (1990) 79–100.

[3] E. Molnár, Polyhedron complexes with simply transitive group actions and their realizations, Acta Math. Hung. 59 (1-2) (1992) 175–216.

[4] E. Molnár, The projective interpretations of the eight 3-dimensional homogeneous geometries, Beiträge zur Algebra und Geometrie (Contributions Alg. Geom.) 38 (2) (1997) 261–288.

[5] E. Molnár, I.Prok, A polyhedron algorithm for finding space groups, Proceedings of Third Int. Conf. On Engineering Graphics and Descriptive Geometry, Vienna 1988, Vol.2, 37–44.

[6] E. Molnár, I. Prok, Classification of solid transitive simplex tilings in simply connected 3-spaces, Part I, Colloquia Math. Soc. János Bolyai 63. Intuitive Geometry, Szeged (Hungary), 1991. North-Holland (1994) 311–362.

[7] E. Molnár, I. Prok, Hyperbolic space form with 'Schläfli solid' (8, 8, 3) Symmetry: Culture and Science, Tessellation 35 (1-2) (2011) 263–267.
[8] E. Molnár, I. Prok, J. Szirmai, Classification of solid transitive simplex tilings in simply connected 3-spaces, Part II, Periodica Math. Hung. 35 (1-2) (1997) 47–94.
[9] E. Molnár, I. Prok, J. Szirmai, Classification of tile-transitive 3-simplex tilings and their realizations in homogeneous spaces, Non-Euclidean Geometries, János Bolyai Memorial Volume, Editors: A. Prékopa and E. Molnár, Mathematics and Its Applications, Vol. 581, Springer (2006) 321–363.
[10] E. Molnár, J. Szirmai, J. Weeks, 3-simplex tilings, splitting orbifolds and manifolds, Symmetry: Culture and Science 22 (3-4) (2011) 435-458.
[11] M. Stojanović, Some series of hyperbolic space groups, Annales Univ. Sci. Budapest, Sect. Math. 36 (1993) 85–102.
[12] M. Stojanović, Hyperbolic realizations of tilings by Zhuk simplices, Matematički Vesnik 49 (1997) 59–68.
[13] M. Stojanović, Hyperbolic space groups for two families of fundamental simplices, Novi Sad J. Math. 29 (3) (1999) 337–348, XII Yugoslav Geometric Seminar, Novi Sad, Octobar 8-11, 1998.
[14] M. Stojanović, Fundamental simplices with outer vertices for hyperbolic groups, Filomat 24 (1) (2010) 1–19.
[15] M. Stojanović, Four series of hyperbolic space groups with simplicial domains, and their supergroups, Krag. J.Math. 35 (2) (2011) 303–315.
[16] M. Stojanović, Supergroups for six series of hyperbolic simplex groups, Periodica Math. Hung. 67 (1) (2013) 115–131.
[17] I. K. Zhuk, Fundamental tetrahedra in Euclidean and Lobachevsky spaces, Soviet Math. Dokl. 270 (3) (1983) 540–543.