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# On Pavlovic's Theorem in Space

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**Abstract** We study higher dimensional counterparts to the well-known theorem of Pavlović (Ann. Acad. Sci. Fenn. Math. **27**, 365–372, 2002), that every harmonic quasiconformal mapping of the disk is bi-Lipschitz.

**Keywords** Bi-Lipschitz maps · Harmonic mappings · Quasiconformal mappings · Higher integrability

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## 1 Introduction

In his influential paper [21] Pavlovic showed that harmonic quasiconformal mappings of the unit disk $\mathbb{D}$ onto itself are bi-Lipschitz mappings. The paper has initiated an extensive

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investigation between the Lipschitz conditions and harmonic quasiconformal mappings, see e.g. [1, 2, 11, 12, 14, 15, 20] and their references.

In this paper we study counterparts of Pavlovic's theorem in higher dimensions.

**Theorem 1.1** *Suppose $f : \mathbb{B}^3 \to \mathbb{B}^3$ is a harmonic quasiconformal mapping, which is also a gradient mapping, that is $f = \nabla u$ for some function $u$ harmonic in the unit ball $\mathbb{B}^3$. Then $f$ is a bi-Lipschitz mapping.*

In two dimensions Pavlovic made a deep and detailed analysis of the boundary values of $f$; analysing them he achieved the Lipschitz-property for every harmonic quasiconformal mapping of the disk. In higher dimensions Pavlovic's approach seems difficult to work with; instead it would seem conceivable that the Lipschitz-property follows by the regularity theory of elliptic PDE's. In fact, such an approach was done by Kalaj [12]. However, the proof in [12] is rather long and technical, and one of the purposes of this note is to give a simple and self-contained argument showing the Lipschitz property in all dimensions.

Thus the main difficulty is to find lower bounds for $|f(x)-f(y)|$ in terms of the distance between $x$ and $y$. In general dimensions it is not even known if harmonic quasiconformal mappings of the ball have all a non-vanishing Jacobian. On the other hand, in three dimensions Lewy [17] proved that for homeomorphic harmonic gradient mappings the Jacobian determinant has no zeroes, and building on this together with work of Gleason and Wolff [7] one arrives at Theorem 1.1.

## 2 Lipschitz Properties in Higher Dimensions

We start with Lipschitz properties for harmonic quasiconformal mappings of the ball, and consider Lipschitz bounds in more general domains in subsequent sections.

**Theorem 2.1** *If $n \geq 2$ and $f : \mathbb{B}^n \to \mathbb{B}^n$ is a harmonic and $K$-quasiconformal mapping, then*

$$|f(x) - f(y)| \leq L|x - y|, \qquad x, y \in \mathbb{B}^n,$$

*where $L$ depends only on the distortion $K$, dimension $n$ and* $\operatorname{dist}(f(0), \mathbb{S}^{n-1})$.

Here and in the sequel $K = K(f)$ denotes the *linear distortion* of $f$, the smallest number $1 \leq K < \infty$ such that the bound

$$\max_{\alpha} |\partial_\alpha f(z)| \leqslant K \min_{\alpha} |\partial_\alpha f(z)|$$

for the directional derivatives of $f$ holds almost everywhere.

For the proof of Theorem 2.1 we only need the Sobolev embedding, which we use in the following local form.

**Lemma 2.1** *Suppose that $w \in W^{2,1}_{loc}(\mathbb{B}^n) \cap C(\overline{\mathbb{B}^n})$, that $h \in L^p(\mathbb{B}^n)$ for some $1 < p < \infty$ and that*

$$\Delta w = h \ \text{ in } \mathbb{B}^n, \ \text{ with } w\big|_{\mathbb{S}^{n-1}} = 0.$$

*a) If $1 < p < n$, then*

$$\|\nabla w\|_{L^q(\mathbb{B}^n)} \leq c(p, n)\|h\|_{L^p(\mathbb{B}^n)}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

*b)* *If* $n < p < \infty$, *then*

$$\|\nabla w\|_{L^\infty(\mathbb{B}^n)} \le c(p,n)\|h\|_{L^p(\mathbb{B}^n)}.$$

The standard proof of Lemma 2.1 follows from the fact that one can represent $w$ in terms of the Green's function $G_{\mathbb{B}^n}(x,y)$ of the unit ball,

$$w(x) = \int_{\mathbb{B}^n} G_{\mathbb{B}^n}(x,y)h(y)dm(y), \qquad x \in \mathbb{B}^n.$$

The Green's function and its gradient

$$\nabla_x G_{\mathbb{B}^n}(x,y) = c_1(n)\left[\frac{x-y}{|x-y|^n} + |y|^n \frac{y-|y|^2 x}{\big|y-|y|^2 x\big|^n}\right]$$

can be explicitly calculated. Since $|y||x-y| \le |y-|y|^2x|$ for all $x, y \in \mathbb{B}^n$, the gradient is bounded by

$$|\nabla_x G_{\mathbb{B}^n}(x,y)| \le 2c_1(n)|y-x|^{1-n} \quad \text{for} \quad x, y \in \mathbb{B}^n.$$

Therefore $\|\nabla w\|_{L^q(\mathbb{B}^n)} \le c\|\mathcal{I}_1 h\|_{L^q(\mathbb{R}^n)}$, where $\mathcal{I}_s h$ denotes the Riesz potential of order $s$. Thus Lemma 2.1.*a*) reduces to the well known boundedness properties of the Riesz potentials,

$$\|\mathcal{I}_s h\|_{L^q(\mathbb{R}^n)} \le c(s,p,q,n)\|h\|_{L^p(\mathbb{R}^n)}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{s}{n},$$

given e.g. in [22, p.119]. The bound in *b*) is easier and follows from Hölder's inequality, since $y \mapsto |x-y|^{1-n} \in L^q(\mathbb{B}^n)$ for every $1 \le q < \frac{n}{n-1}$.

The above local form of Sobolev's embedding yields a quick proof for the following

**Corollary 2.1** *Suppose* $w \in W^{2,1}_{loc}(\mathbb{B}^n) \cap C(\overline{\mathbb{B}^n})$, $n \ge 2$, *is such that*

$$w\big|_{\mathbb{S}^{n-1}} = 0, \quad \textit{with} \quad \int_{\mathbb{B}^n} |\nabla w|^{p_0}\, dm < \infty \ \textit{for some}\ n < p_0 < \infty. \tag{2.1}$$

*If* $w$ *satisfies the following uniform differential inequality,*

$$|\Delta w(x)| \le a|\nabla w(x)|^2 + b, \qquad x \in \mathbb{B}^n, \tag{2.2}$$

*for some constants* $a, b < \infty$, *we then have*

$$\|\nabla w\|_{L^\infty(\mathbb{B}^n)} \le M < \infty, \tag{2.3}$$

*where* $M = M(a, b, p_0, n, \|\nabla w\|_{p_0})$. *In particular,* $w$ *is Lipschitz continuous.*

*Proof* According to Eq. 2.2 we have

$$\Delta w(x) = h(x), \qquad \text{for a.e.} x \in \mathbb{B}^n, \tag{2.4}$$

where

$$h(x) = c(x)\left(|\nabla w(x)|^2 + 1\right) \tag{2.5}$$

and $\|c\|_\infty \le \max\{a,b\}$; one can simply define $c(x) := \Delta w(x)\left(|\nabla w(x)|^2+1\right)^{-1}$ for almost every $x \in \mathbb{B}^n$.

Here by our assumptions $\nabla w \in L^{p_0}(\mathbb{B}^n)$. However, with Sobolev embedding one can improve this integrability, up to

$$\nabla w \in L^s(\mathbb{B}^n) \qquad \text{where } s > 2n. \tag{2.6}$$

Indeed, if $p_0/2 < n < p_0$, then $h(x) = c(x)\left(|\nabla w(x)|^2 + 1\right) \in L^{p_0/2}(\mathbb{B}^n)$, and (2.4) with Lemma 2.1 *a*) give

$$\nabla w \in L^{p_1}(\mathbb{B}^n), \qquad p_1 = \frac{p_0 n}{2n - p_0} > p_0, \tag{2.7}$$

which is a strict improvement in the integrability.

To quantify this, note that if initially $p_0 = (1+\varepsilon)n$, $\varepsilon > 0$, then

$$p_1 = \frac{p_0 n}{2n - p_0} = \frac{1+\varepsilon}{1-\varepsilon} n > (1 + 2\varepsilon)n$$

Thus one can iterate this feedback argument, getting $\nabla w \in L^{p_\ell}(\mathbb{B}^n)$, $\ell = 0, 1, 2, \dots$ with $p_\ell > (1 + 2^\ell \varepsilon)n$, until the condition (2.6) is reached. (If it happens that for some exponent $p_\ell = 2n$, we can choose $p_0$ little smaller so that this degeneracy does not happen.)

And once Eq. 2.6 is achieved, (2.4)-(2.5) with Sobolev embedding, Lemma 2.1 b), give $\|\nabla w\|_\infty < \infty$. The proof also gives a bound for $\|\nabla w\|_\infty$ that depends only on the constants $a$ and $b$, the exponent $p_0$ and the initial norm $\|h\|_{p_0/2} \le \max\{a, b\}(\|\nabla w\|_{p_0}^2 + 1)$. □

*Remark 2.1* It is interesting to note that the above iteration argument fails if in Eq. 2.1 one assumes integrability only for some $1 \le p_0 < n$. Thus higher integrability, and Gehring's theorem [6] in case of quasiconformal mappings, become particularly useful also here.

*Remark 2.2* One can replace the zero boundary values in Eq. 2.1 e.g. by the requirement $w\big|_{\mathbb{S}^{n-1}} \in C^{1,\alpha}$, by considering $w - P[w]$, where $P[w]$ is the Poisson integral of $w$. Similarly, by properties of the Green's function the conclusions can be improved to $\|\nabla w\|_{L^\infty(\mathbb{B}^n)} + \|w\|_{C^{1,\alpha}(\mathbb{B}^n)} \le M < \infty,\ 0 < \alpha < 1$.

We can now turn to proving the Lipschitz bounds for harmonic $K$-quasiconformal mappings $f = (f^1, \dots, f^n) : \mathbb{B}^n \to \mathbb{B}^n$. For this note that the mere harmonicity gives

$$\Delta(f^j)^2(x) = 2|\nabla f^j(x)|^2, \qquad j = 1, \dots, n.$$

Thus any "reasonable" function of $f^1, \dots, f^n$ will satisfy the differential inequality (2.2). To get uniform Lipschitz bounds, we need in addition some normalisation such as vanishing on the boundary $\mathbb{S}^{n-1}$, like in Eq. 2.1. Therefore a convenient choice for our purposes is e.g. $w(x) = 1 - |f(x)|^2$.

*Proof of Theorem 2.1* We first recall Gehrings famous theorem [6] which gives for every quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ the higher integrability

$$\int_{\mathbb{B}^n} |Df(x)|^p dm \le C < \infty, \qquad p = p(n, K) > n, \tag{2.8}$$

where for mappings of the whole space $\mathbb{R}^n$, the constant $C$ depends only on $n$ and distortion $K(f)$.

In case $f : \mathbb{B}^n \to \mathbb{B}^n$ is $K$-quasiconformal, we can compose $f$ with a Möbius transform $\psi$ preserving the ball, such that $f \circ \psi(0) = 0$. With Schwarz reflection one can then extend $f \circ \psi$ to $\mathbb{R}^n$ and apply Eq. 2.8 to this mapping. Unwinding the Möbius transform, i.e. after a change of variables, we see that any $K$-quasiconformal mapping $f : \mathbb{B}^n \to \mathbb{B}^n$ satisfies (2.8) with $C = C(n, K, \mathrm{dist}(f^{-1}(0), \mathbb{S}^{n-1}))$

If in addition $f$ is harmonic, consider the function

$$w(x) = 1 - |f(x)|^2, \qquad x \in \mathbb{B}^n.$$

Since quasiconformal mappings of $\mathbb{B}^n$ extend continuously to the boundary, $w(x)$ satisfies the assumptions of Corollary 2.1. For the condition (2.2) note that $w = u \circ f$ where

$$u(x) = 1 - |x|^2 \quad \text{with} \quad \nabla u(x) = -2x, \qquad x \in \mathbb{B}^n.$$

Thus $\nabla w(x) = Df^t(x)\nabla u(f(x))$, so that

$$\frac{2}{K}|f(x)|\,|Df(x)| \leq |\nabla w(x)| \leq 2|f(x)|\,|Df(x)| \tag{2.9}$$

with

$$|\Delta w(x)| = 2||Df(x)||^2 \leq 2n^2|Df(x)|^2, \qquad x \in \mathbb{B}^n.$$

where $||Df(x)||^2$ denotes the Hilbert-Schmidt norm of the differential matrix.

The above already establishes Eq. 2.2. However, to see the explicit dependence of $a$ and $b$ on properties of the mapping $f$ we first note that there is a constant $\delta = \delta\big(n, K, \text{dist}(f(0), \mathbb{S}^{n-1})\big)$ such that

$$1 - |x| + |f(x)| \geq \delta > 0, \qquad \text{for all} x \in \mathbb{B}^n. \tag{2.10}$$

Indeed, quasiconformal mappings of $\mathbb{B}^n$ are rough isometries in the hyperbolic metric [23]. Thus either $h_{\mathbb{B}^n}\big(f(x), f(0)\big) \geq 1 + h_{\mathbb{B}^n}\big(f(0), 0\big) \Rightarrow h_{\mathbb{B}^n}\big(f(x), 0\big) \geq 1$ and $|f(x)| \geq \frac{e-1}{e+1}$, or else we have $h_{\mathbb{B}^n}(x, 0) \leq c(K)(1 + h_{\mathbb{B}^n}\big(f(x), f(0)\big)) \leq c(K)(2 + h_{\mathbb{B}^n}\big(f(0), 0\big))$. In the latter case, $1 - |x| \geq e^{-M}$ where $M := c(K)(2 + h_{\mathbb{B}^n}\big(f(0), 0\big))$. Thus Eq. 2.10 holds, and we have

$$|\Delta w| \leq \frac{4n^2}{\delta^2}[(1 - |x|)^2 + |f(x)|^2]|Df(x)|^2 \leq \frac{K^2n^2}{\delta^2}|\nabla w|^2 + \frac{4n^2}{\delta^2}(1 - |x|)^2|Df(x)|^2.$$

The last term is controlled by basic ellipticity bounds [10, p.38], i.e. the Bloch norm bounds

$$(1 - |x|)|Df(x)| \leq c(n)\|f\|_\infty \tag{2.11}$$

valid for every harmonic function. Thus (2.2) holds with $a = K^2n^2\delta^{-2}$, $b = 4n^2c(n)^2\delta^{-2}$, so that $\nabla w \in L^\infty(\mathbb{B}^n)$ by Corollary 2.1.

Similarly (2.9)–(2.11) give

$$|Df(x)| \leq \frac{1}{\delta}(1 - |x| + |f(x)|)|Df(x)| \leq \frac{c(n)}{\delta} + \frac{K\|\nabla w\|_\infty}{2\delta},$$

so that $f$ is a Lipschitz mapping, with an explicit bound $L = L\big(n, K, \text{dist}(f(0), \mathbb{S}^{n-1})\big)$ for its Lipschitz constant. □

## 3 Co-Lipschitz Mappings

We say that a mapping $f$ defined in a domain $\Omega \subset \mathbb{R}^n$ has the co-Lipschitz property with constant $1 \leq L$, if

$$|f(x) - f(y)| \geq \frac{1}{L}|x - y| \qquad \forall\, x, y \in \Omega. \tag{3.1}$$

The inverse of a $K$-quasiconformal mapping is also $K$-quasiconformal mapping, but for harmonic $f$ the inverse $f^{-1}$ is not in general harmonic. Hence even for harmonic quasiconformal mappings of the ball, the co-Lipschitz property does not follow from Theorem 2.1.

Naturally, for mappings in Eq. 3.1 the Jacobians are non-vanishing everywhere. In dimensions $n \geq 3$, the Jacobian of a harmonic homeomorphism may vanish, see e.g. [5, p.26], and therefore the co-Lipschitz property is a more subtle problem than in dimension $n = 2$.

On the other hand, for quasiconformal mappings we have the following geometric notion of an average derivative, see [3, Definition 1.5],

$$\alpha_f(x) = \exp\left(\frac{1}{n}(\log J_f)_{B_x}\right). \tag{3.2}$$

Here

$$(\log J_f)_{B_x} = \frac{1}{m(B_x)}\int_{B_x} \log J_f\, dm, \qquad B_x = B(x, d(x, \partial\Omega)).$$

Since for a quasiconformal mapping, the Jacobian $J_f$ is an $A_\infty$-weight, $\alpha_f(x)$ is comparable to $\left(\frac{1}{m(B_x)}\int_{B_x} J_f^p\right)^{1/p}$ for every $0 < p \le 1$, and hence we could have used such averages as well. On the other hand, in the case $n = 2$ and $f$ conformal we have

$$\alpha_f(z) = |f'(z)|$$

and therefore the above choice (3.2) appears a natural one. Furthermore, we have the following quasiconformal version of the Koebe-distortion theorem, see [3, Theorem 1.8].

**Theorem 3.1** *Suppose that $\Omega$ and $\Omega'$ are domains in $\mathbb{R}^n$. If $f : \Omega \longrightarrow \Omega'$ is $K$-qc, then*

$$\frac{1}{c}\,\frac{d(f(x), \partial\Omega')}{d(x, \partial\Omega)} \le \alpha_f(x) \le c\,\frac{d(f(x), \partial\Omega')}{d(x, \partial\Omega)}$$

*for $x \in \Omega$, where $c$ is a constant which depends only on $K$ and $n$.*

As a step towards the co-Lipschitz properties we next need a version of the classical Hopf lemma, see e.g. [4, Exercise I.25]. This is often applied in various contexts for estimating superharmonic functions in terms of boundary distance. For example, Mateljević used this well known approach for harmonic quasiconformal mappings in [19].

**Theorem 3.2** *Suppose $f : \mathbb{B}^n \longrightarrow \Omega$ is a harmonic quasiconformal mapping, with $\Omega \subset \mathbb{R}^n$ a convex subdomain. Then*

$$\alpha_f(x) \ge c_0\, d(f(0), \partial\Omega) > 0, \qquad x \in \mathbb{B}^n, \tag{3.3}$$

*where the constant $c_0 = c_0(n, K)$ depends only on the dimension $n$ and distortion $K = K(f)$.*

*Proof* For every $x \in \mathbb{B}^n$ we have

$$d(f(x), \partial\Omega) = \inf_p d(f(x), p),$$

where infimum is taken over all hyperplanes $p$ outside the domain. Since $d(f(x), p) = \langle f(x), n\rangle + \mathit{const}.$, where $n$ is a normal to $p$, the function $x \mapsto d(f(x), p)$ is positive and harmonic in $\mathbb{B}^n$. We denote this function by $h_p(x)$, and for each $h_p$ apply the usual Harnack inequality in $\mathbb{B}^n$,

$$h_p(x) \ge \frac{1 - |x|}{(1 + |x|)^{n-1}}\, h_p(0).$$

Because $d(f(0), p) \ge d(f(0), \partial\Omega)$ we have

$$h_p(x) \ge \frac{1 - |x|}{(1 + |x|)^{n-1}}\, d(f(0), \partial\Omega).$$

Infimum of the last inequality over all $p$ gives

$$d(f(x), \partial\Omega) \geq \frac{1-|x|}{(1+|x|)^{n-1}} d(f(0), \partial\Omega).$$

Finally, as

$$d(x, \partial\mathbb{B}^n) = 1 - |x|$$

the last inequality we can write as

$$\frac{d(f(x), \partial\Omega)}{d(x, \partial\mathbb{B}^n)} \geq \frac{d(f(0), \partial\Omega)}{(1+|x|)^{n-1}}.$$

Using then Theorem 3.1 and quasiconformality of $f$ we conclude that

$$\alpha_f(x) \geq c(n, K) d(f(0), \partial\Omega).$$

Thus one can achieve the co-Lipschitz property if the usual derivative can be estimated from below by the average derivative. In two dimensions this can be done by the next key result of the second author, see [18].

**Theorem 3.3** *Suppose $\Omega, \Omega' \subset \mathbb{R}^2$ are planar domains and $f : \Omega \to \Omega'$ a harmonic quasiconformal mapping. Then* $\log J_f$ *is superharmonic in $\Omega$.* □

Now, we can use the superharmonicity of $\log J_f$ for the harmonic quasiconformal mapping $f$ defined in the unit disk $\mathbb{B}^2$,

$$\log|Df(z)|^2 \geq \log J_f(z) \geq \frac{1}{m(B_z)} \int_{B_z} \log J_f \, dm = \log \alpha_f(z)^2, \qquad z \in \mathbb{B}^2. \tag{3.4}$$

This estimate combined with Theorem 3.2 proves for every harmonic quasiconformal mapping from the disk onto a convex domain the lower bound

$$\inf_{|h|=1} |Df(z)h| \geq |Df(z)|/K \geq \alpha_f(z)/K \geq c d(f(0), \partial\Omega) \tag{3.5}$$

for some constant $c > 0$. From this we can conclude that $f$ is co-Lipschitz. This gives a new proof for [13, Cor 2.7]. In fact we have more generally,

**Corollary 3.1** *Suppose $\Omega, \Omega' \subset \mathbb{R}^2$ are simply connected domains and $f : \Omega \to \Omega'$ is a harmonic quasiconformal mapping. If $\Omega'$ is convex and the Riemann map of $\Omega$ has derivative bounded from above, the $f$ has the coLipschitz property* (3.1).

The proof follows by applying (3.5) to $f \circ g$, where $g : \mathbb{D} \to \Omega$ is the Riemann map. So in particular, in Corollary 3.1 the boundary of $\Omega$ need not be $C^1$, not even Lipschitz. For instance, $g(z) = 2z - z^2$ is a conformal map from $\mathbb{D}$ onto a cardioid, with cusp at $1 = g(1)$.

Similarly, combining (3.4) with Theorems 2.1 and 3.2 we have a new proof for Pavlovic's theorem in $\mathbb{B}^2$.

To complete the proof of Theorem 1.1 we use an argument analogous to Eq. 3.4 and Theorem 3.3. First, by Lewy's theorem [17], if the gradient $f = \nabla u$ of a (real valued) harmonic function defines a homeomorphism $f : \Omega \to \Omega'$ where $\Omega, \Omega' \subset \mathbb{R}^3$, then the Jacobian $J_f$ does not vanish. Further, $J_f = \mathcal{H}_u$, where $\mathcal{H}_u$ denotes the Hessian of $u$, and here we have the theorem of Gleason and Wolff [7, Theorem A] that again in dimension three, the function $\log\det(\mathcal{H}_u)$ is superharmonic outside the zeroes of the Hessian. We collect these facts in the following

**Theorem 3.4** (Lewy-Gleason-Wolff) *Suppose $u : \Omega \to \mathbb{R}$ is a harmonic function, such that $f(x) := \nabla u(x)$ defines a homeomorphism between the domains $\Omega$ and $\Omega' \subset \mathbb{R}^3$. Then*

$$\log J_f(z) = \log\det(\mathcal{H}_u) \ \ \textit{is superharmonic in}\, \Omega. \tag{3.6}$$

With these arguments we finally have a proof of Theorem 1.1. Indeed, if $f = \nabla u$ is a quasiconformal harmonic gradient mapping in $\mathbb{B}^3$ then as in Eq. 3.4, using Theorem 3.4 we conclude that

$$\alpha_f(x)^3 \le J_f(x) \le K(f)^2 \inf_{|h|=1} |Df(x)h|^3.$$

Thus when $f(\mathbb{B}^3) = \mathbb{B}^3$, or more generally when the target domain is convex, Theorem 3.2 gives the co-Lipschitz property for $f$. The Lipschitz-properties follow from Theorem 2.1, completing the proof of Theorem 1.1. □

## 4 Harmonic Quasiconformal Mappings in General Domains

The above approach to higher dimensional Pavlovic's theorem has a few consequences also in more general subdomains of $\mathbb{R}^n$. To discuss these, we start with the quasihyperbolic metric introduced by Gehring and Palka in [9]. In this work they used the metric as a tool to understand quasiconformal homogeneity. The metric has since been studied by number of different authors.

**Definition 4.1** Let $D$ be a proper subdomain of $\mathbb{R}^n$, $n \ge 2$. We define the quasihyperbolic length of a rectifiable curve $\gamma \subset D$ by

$$l_k(\gamma) = \int_\gamma \frac{ds}{d(x, \partial D)}.$$

The quasihyperbolic metric is defined by

$$k_D(x_1, x_2) = \inf_\gamma (l_k(\gamma)),$$

where the infimum is taken over all rectifiable curves in $D$ joining $x_1$ and $x_2$.

Quasihyperbolic metric is invariant under Euclidean isometries and homoteties but it is not invariant under conformal mappings, it is not even Möbius invariant. By a result of Gehring and Osgood [8], for any domain $D \subseteq \mathbb{R}^n$ and points $x, y \in D$ there exists a quasihyperbolic geodesic. Moreover, quasihyperbolic metric is quasi-invariant under conformal and more generally under quasiconformal mappings. Namely, there is a constant $0 < C = C(n, K) < \infty$ such that

$$k_{D'}(f(x_1), f(x_2)) \le C \cdot \max\{k_D(x_1, x_2), k_D^\alpha(x_1, x_2)\}, \quad \alpha = K^{1/(1-n)},$$

for all $x_1, x_2 \in D$, whenever $f$ is a $K$-quasiconformal mapping from $D$ onto $D'$.

If we deal with harmonic quasiconformal mappings between two general proper subdomains of $\mathbb{R}^2$, then such mappings are bi-Lipschitz with respect to corresponding quasihyperbolic metrics [18]. Here we have a generalization of this result in space.

**Theorem 4.1** *Consider domains $\Omega, \Omega' \subset \mathbb{R}^3$, and let $f : \Omega \to \Omega'$ be a quasiconformal homeomorphism. If one can represent $f(x) = \nabla u(x)$ as the gradient of some function $u$*

*harmonic in* $\Omega$*, then* $f$ *is bi-Lipschitz with respect to the corresponding quasihyperbolic metrics,*

$$\frac{1}{M}k_\Omega(x, y) \le k_{\Omega'}(f(x), f(y)) \le Mk_\Omega(x, y), \qquad x, y \in \Omega,$$

*where the constant* $M$ *depends only on the distortion* $K(f)$*.*

*Proof* From (3.4) and Theorem 3.4 we get $\alpha_f(x)^3 \le J_f(x)$, $x \in \Omega$. On the other hand, $Df(x)$ is a vector valued harmonic function, whose norm is subharmonic and thus

$$\|Df(x)\|^3 \le \frac{1}{m(B_x)}\int_{B_x}\|Df\|^3\,dm \le \frac{K}{m(B_x)}\int_{B_x} J_f\,dm$$
$$\le C(K, n)\exp[\frac{1}{m(B_x)}\int_{B_x}\log J_f\,dm] = C(K, n)\alpha_f(x)^3,$$

where the third inequality follows from the fact that $J_f$ is an $A_\infty$-Muckenhoupt weight. Thus $\alpha_f(x) \simeq \inf_{|h|=1}|Df(x)h| \simeq \sup_{|h|=1}|Df(x)h|$, and the claim follows as in [18]. □

The proof of Theorem 1.1 at the end of the previous section gives immediately

**Corollary 4.1** *Suppose* $f : \mathbb{B}^3 \to \Omega$ *is quasiconformal. If* $\Omega$ *is convex and* $f = \nabla u$ *is the gradient of a harmonic function, then* $f$ *has the co-Lipschitz property* (3.1).

Similarly, the method of Theorem 2.1. works for more general domains. We have the following result of Kalaj [12].

**Corollary 4.2** *If* $f : \mathbb{B}^n \to \Omega$ *is a harmonic quasiconformal mapping, where* $\Omega \subset \mathbb{R}^n$ *is a domain with* $C^2$*-boundary, then* $f$ *is a Lipschitz mapping.*

*Proof* We take this time $w(x) = \text{dist}(f(x), \partial\Omega)$ near $\partial\Omega$, and choose some smooth extension to $\Omega$. Then $w$ satisfies the inequality (2.2), see [12], so that $\|\nabla w\|_\infty < \infty$ by Corollary 2.1 and we obtain the Lipschitz bounds for $f$ as in the proof of Theorem 2.1. □

Collecting the above information we also have

**Theorem 4.2** *Suppose* $\Omega$ *is a convex subdomain of* $\mathbb{R}^3$ *with* $C^2$*-boundary, and let* $f : \mathbb{B}^3 \to \Omega$ *be a quasiconformal homeomorphism. If* $f = \nabla u$ *is a harmonic gradient mapping then* $f$ *is bi-Lipschitz, with respect to the Euclidean metric.*

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## References

1. Arsenović, M., Bozin, V., Manojlović, V.: Moduli of continuity of harmonic quasiregular mappings in $\mathbb{B}^n$. Potential Anal. **34**, 283–291 (2011)
2. Arsenović, M., Manojlović, V., Mateljević, M.: Lipschitz-type spaces and harmonic mappings in the space. Ann. Acad. Sci. Fenn., Math. **35**, 379–387 (2010)
3. Astala, K., Gehring, F.W.: Quasiconformal analogues of theorems of Koebe and Hardy-Littlewood. Mich. Math. J. **32**, 99–107 (1985)
4. Axler, S., Bourdon, P., Ramey, W.: Harmonic Function Theory. Springer Verlag New York (1992)

5. Duren, P.: Harmonic Mappings in the Plane, Cambridge Tract in Mathematics 156. Cambridge University Press (2004)
6. Gehring, F.W.: The $L^p$-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973)
7. Gleason, S., Wolff, T.: Lewy's harmonic gradient maps in higher dimensions Comm. Partial Diff. Equat. **16**(12), 1925–1968 (1991)
8. Gehring, F.W., Osgood, B.: Uniform domains and the quasihyperbolic metric. J. Anal. Math. **36**, 50–74 (1979)
9. Gehring, F.W., Palka, B.: Quasiconformally homogeneous domains. J. Anal. Math. **30**, 172–199 (1976)
10. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Springer Verlag (1998)
11. Kalaj, D.: Quasiconformal harmonic mapping between Jordan domains. Math. Z. **260**, 237–252 (2008)
12. Kalaj, D.: A priori estimate of gradient of a solution to certain differential inequality and quasiconformal mappings. J. D'Anal. Math. **119**, 63–88 (2013)
13. Kalaj, D.: On harmonic diffeomorphisms of the unit disk onto a convex domain. Complex Variables **48**, 175–187 (2003)
14. Kalaj, D., Pavlović, M.: Boundary correspondence under quasiconformal harmonic diffeomorphisms of a half-plane. Ann. Acad. Sci. Fenn. Math. **30**, 159–165 (2005)
15. Kalaj, D., Mateljević, M.: Inner estimate and quasiconformal harmonic maps between smooth domains. J. D'Anal. Math. **100**, 117–132 (2006)
16. Kalaj, D., Saksman, E.: Quasiconformal maps with controlled Laplacian. Mathematics. arXiv:1410.8439 (2014)
17. Lewy, H.: On the Non-Vanishing of the Jacobian of a Homeomorphism by Harmonic Gradients. Ann. Math. **88**, 518–529 (1968)
18. Manojlović, V.: Bi-Lipschicity of Quasiconformal Harmonic Mappings in the Plane. Filomat **23**, 85–89 (2009)
19. Mateljević, M.: Distortion of harmonic functions and harmonic quasiconformal quasi-isometry, Revue Roum. Math. Pures Appl. **51**(5-6), 711–722 (2006)
20. Mateljević, M., Vuorinen, M.: On harmonic quasiconformal quasi-isometries, Journal of Inequalities and Applications (2010)
21. Pavlović, M.: Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disc. Ann. Acad. Sci. Fenn. Math. **27**, 365–372 (2002)
22. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press (1970)
23. Vuorinen, M.: Conformal Geometry and quasiregular mappings, Lecture Notes in Mathematics, vol. 1319. Springer Verlag, Berlin (1988)