МАТЕМАТИЧКИ ВЕСНИК
54 (2002), 125–131

UDK 517.927
оригинални научни рад
research paper

# AN $L_p$ ESTIMATE FOR THE DIFFERENCE OF DERIVATIVES OF SPECTRAL EXPANSIONS ARISING BY ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

**Nebojša L. Lažetić and Olivera R. Đorđević**

**Abstract.** We prove the estimate

$$\|\sigma'_\mu(x,f)-\tilde{\sigma}'_\mu(x,f)\|_{L_p(G)}\le C\|f\|_{BV(G)}\cdot\mu^{1-1/p},$$

where $2\le p<+\infty$, and $\sigma_\mu(x,f),\tilde{\sigma}_\mu(x,f)$ are the partial sums of spectral expansions of a function $f(x)\in BV(G)$, corresponding to arbitrary non-negative self-adjoint extensions of the operators $\mathcal{L}u=-u''+q(x)u$, $\tilde{\mathcal{L}}u=-u''+\tilde{q}(x)u$ $(x\in G)$ respectively; the operators are defined on an arbitrary bounded interval $G\subset\mathbb{R}$.

## 1. Introduction

Let $G=(a,b)$ be an arbitrary bounded interval, and let the operators

$$\mathcal{L}u=-u''+q(x)u,\qquad \tilde{\mathcal{L}}u=-u''+\tilde{q}(x)u \tag{1}$$

be defined on $G$, with potentials $q(x),\tilde{q}(x)\in L_s(G)$, $1<s\le 2$. Denote by $L,\tilde{L}$ arbitrary non-negative self-adjoint extensions, with discrete spectrum, of the operators (1) respectively (see §17, [1]). Let $\{u_n(x)\}_{n=1}^\infty,\{\tilde{u}_n(x)\}_{n=1}^\infty$ be complete (in $L_2(G)$) and orthonormal systems of eigenfunctions of those extensions, and $\{\lambda_n\}_{n=1}^\infty,\{\tilde{\lambda}_n\}_{n=1}^\infty$ the corresponding systems of non-negative eigenvalues, enumerated in non-decreasig order. If $f(x)\in L_1(G)$ and $\mu\ge 2$, we can form the partial sums of order $\mu$:

$$\sigma_\mu(x,f)=\sum_{\sqrt{\lambda_n}<\mu} f_n u_n(x),\qquad \tilde{\sigma}_\mu(x,f)=\sum_{\sqrt{\tilde{\lambda}_n}<\mu}\tilde{f}_n\tilde{u}_n(x),$$

where $f_n=\int_a^b f(x)u_n(x)\,dx$, $\tilde{f}_n=\int_a^b f(x)\tilde{u}_n(x)\,dx$. Let $AC(G)$ be the set of absolutely continuous functions on the closed interval $\overline{G}$. Denote by $BV(G)$ the Banach space of functions having bounded variation on $\overline{G}$, with the norm $\|f\|_{BV(G)}=\sup_{x\in\overline{G}}|f(x)|+V_a^b(f)$, where $V_a^b(f)$ stands for the total variation of $f(x)$ on $\overline{G}$.

*AMS Subject Classification*: 34 L 10, 47 E 05

Communicated at the 5th International Symposium on Mathematical Analysis and its Applications, Niška banja, Yugoslavia, October, 2–6, 2002.

The problem of behavior of function $\sigma_\mu(x,f)$ (and its derivatives) on subsets of $\overline{G}$, as $\mu \to +\infty$, is the classical one. One of the most fruitful approaches to the problem is so-called "equiconvergence approach": one studies the behavior of the difference $\sigma_\mu(x,f) - S_\mu(x,f)$, as $\mu \to +\infty$, where $S_\mu(x,f)$ is the corresponding partial sum of the trigonometrical Fourier series of function $f$ (for a review see [2]). It seems that the first results concerning the equiconvergence rate estimates, in the case of arbirtary self-adjoint Sturm-Liouville operators, were obtained by V.A. Il'in and I. Joo in [3]. They obtained the following estimate:

If $q(x), \tilde{q}(x) \in L_s(G)$ $(s > 1)$, $f(x) \in AC(G)$, and $K \subset G$ is an arbitrary compact set, then there exists a constant $C(K,f) > 0$ such that

$$\max_{x \in K} |\sigma_\mu(x,f) - \tilde{\sigma}_\mu(x,f)| \le C(K,f) \cdot \frac{1}{\mu}, \qquad \mu \ge 2; \tag{2}$$

$C(K,f)$ does not depend on $\mu$. The estimate is exact in order with respect to $\mu$.

In order to "globalize" the estimate (2), I.S. Lomov has considered the $L_p$ metric instead of the uniform one; in paper [4] he proved the following assertion: If $q(x), \tilde{q}(x) \in L_s(G)$ $(s > 1)$, $f(x) \in BV(G)$, and $2 \le p < +\infty$, then the estimate

$$\|\sigma_\mu(x,f) - \tilde{\sigma}_\mu(x,f)\|_{L_p(G)} \le C\|f\|_{BV(G)} \cdot \frac{1}{\mu^{1/p}}, \qquad \mu \ge 3, \tag{3}$$

holds, where $C > 0$ does not depend on $f$ and $\mu$. (Note that in earlier paper [5] Lomov obtained estimate (3) with $\mu^{-1/p} \ln \mu$ instead of $\mu^{-1/p}$.)

A local uniform estimate for the difference of the first derivatives $\sigma'_\mu(x,f)$, $\tilde{\sigma}'_\mu(x,f)$ was obtained by I. Joo and N. Lažetić in paper [6]. They proved: If $q(x)$ and $\tilde{q}(x)$ belong to $L_s(G)$ $(1 < s \le 2)$, $f(x) \in AC(G)$, and $K \subset G$ is an arbitrary compact set, then the estimate

$$\max_{x \in K} |\sigma'_\mu(x,f) - \tilde{\sigma}'_\mu(x,f)| \le C(K,f), \qquad \mu \ge 2, \tag{4}$$

holds, where $C(K,f) > 0$ is independent of $\mu$. This estimate is exact in order with respect to the spectral parameter $\mu$.

Recently, the estimate (4) has been extendend on the set $BV(G)$. Namely, the authors of this paper have proved ([7]) that for every function $f(x) \in BV(G)$ and every compact set $K \subset G$ the following estimate is valid:

$$\max_{x \in K} |\sigma'_\mu(x,f) - \tilde{\sigma}'_\mu(x,f)| \le C(K)\|f\|_{BV(G)}, \qquad \mu \ge 2. \tag{5}$$

It is supposed that $q(x), \tilde{q}(x) \in L_s(G) (s > 1)$.

In this paper we propose an $L_p$ estimate for the difference mentioned above. That estimate "globalizes" (5), and shows how the estimate (3) is affected by the operation of differentiation (compare with estimates (8)-(9) below). Hence, our result is the following assertion.

THEOREM. *Suppose $q(x), \tilde{q}(x) \in L_s(G) (1 < s \le 2), f(x) \in BV(G)$, $p \in [2, +\infty)$, and $\mu \ge 2$. There exists a constant $C > 0$, independent of $f$ and $\mu$, such that the following estimate holds:*

$$\|\sigma'_\mu(x,f) - \tilde{\sigma}'_\mu(x,f)\|_{L_p(G)} \le C\|f\|_{BV(G)} \cdot \mu^{1-1/p}. \tag{6}$$

## 2. Auxiliary results

Proof of the theorem is based on estimate (5). But we will also use a variety of known results listed below.

Let $q(x) \in L_1(G)$. Then for systems of eigenfunctions and eigenvalues of an arbitrary non-negative self-adjoint extension $L$ of the operator $\mathcal{L}$ the following estimates are valid:

$$\sum_{|\sqrt{\lambda_n}-\mu|<1} 1 \leq A, \qquad \mu > 0, \tag{7}$$

where $A > 0$ does not depend on $\mu$ (see [8] and [9]);

$$\sup_{x\in G} |u_n(x)| \leq C(G), \tag{8}$$

where $C(G) > 0$ is independent of $n \in \mathbb{N}$ ([8]);

$$\sup_{x\in G} |u_n'(x)| \leq C_1(G)(\sqrt{\lambda_n}+1), \tag{9}$$

with $C_1(G) > 0$ non-depending on $n \in \mathbb{N}$ ([10]).

If $f(x) \in BV(G)$, then for its Fourier coefficients $f_n$ (with respect to the system $\{u_n(x)\}_{n=1}^{\infty}$) the estimate

$$|f_n| \leq \frac{C}{\sqrt{\lambda_n}} \cdot \|f\|_{BV(G)} \tag{10}$$

holds, where $C > 0$ does not depend on $n \in \mathbb{N}$ (see [5]).

We will also use so-called "mean value formula" for the derivatives $u_n'(x)$ ([10]): If $x \in G$ and $t > 0$ are such that $x \pm t \in G$, then

$$u_n'(x+t) - u_n'(x-t) = -2\sqrt{\lambda_n}u_n(x)\sin\sqrt{\lambda_n}t + \\ + \int_{x-t}^{x+t} q(\xi)u_n(\xi)\cos\sqrt{\lambda_n}(|x-\xi|-t)\,d\xi. \tag{11}$$

(Note that a function $u_\lambda(x)$ is called an eigenfunction corresponding to an eigenvalue $\lambda$ of the operator $L$ if $u_\lambda(x), u_\lambda'(x) \in AC(G)$ and the equality

$$-u_\lambda''(x) + q(x)u_\lambda(x) = \lambda u_\lambda(x)$$

holds a.e. on $G$.)

Finally, recall the "second part" of the known Riesz theorem ([11]): Let $\{v_n(x)\}_{n=1}^{\infty}$ be an orthogonal system of functions defined on a bounded interval $G$, and such that $\sup_{x\in G}|v_n(x)| \leq M$, where $M > 0$ is independent on $n \in \mathbb{N}$. If $1 < r \leq 2$ and $1/r + 1/p = 1$, then for every sequence of (complex) numbers $\{g_n\}_{n=1}^{\infty}$, satisfying $\left(\sum_{n=1}^{\infty}|g_n|^r\right)^{1/r} < +\infty$, there exists a function $g(x) \in L_p(G)$ such that $g_n = \int_a^b g(y)\overline{v_n(y)}\,dy$ and

$$\|g\|_{L_p(G)} \leq M^{2/r-1}\Big(\sum_{n=1}^{\infty}|g_n|^r\Big)^{1/r}. \tag{12}$$

Note that in proving the estimate (5) we have used all the results (7)–(12).

## 3. Proof of the theorem

The first step of the proof is the same as the one in the proof of Lemma 2 [5]. Let $K=[c,d]\subset G$ be an arbitrary fixed closed interval. Then we have

$$\|\sigma'_\mu(x,f)-\tilde{\sigma}'_\mu(x,f)\|^p_{L_p(G)}=\|(\cdot)\|^p_{L_p((a,c))}+\|(\cdot)\|^p_{L_p(K)}+\|(\cdot)\|^p_{L_p((d,b))}. \tag{13}$$

In estimating the members on the right-hand side, we will assume, with no loss of generality, that $\lambda_n\geq 1$ ($n\in\mathbb{N}$). (This assumption is based on the equation $-u''_n(x)+[q(x)+1]u_n(x)=(\lambda_n+1)u_n(x)$.) Set $\mu_n\overset{\text{def}}{=}\sqrt{\lambda_n}$.

Let us consider the first member. Introducing a new variable $z=x+h$, with $h\in(0,(d-c)/2)$ fixed, we obtain

$$\|\sigma'_\mu(x,f)-\tilde{\sigma}'_\mu(x,f)\|_{L_p((a,c))}=\int_{K_1}|\sigma'_\mu(z-h,f)-\tilde{\sigma}'_\mu(z-h,f)|^p\,dz, \tag{14}$$

where $K_1\overset{\text{def}}{=}[a+h,c+h]\subset G$. By the mean value formula (11), we can write

$$\begin{aligned}\sigma'_\mu(z-h,f)=&\sum_{\mu_n<\mu}f_nu'_n(z+h)+\sum_{\mu_n<\mu}2\mu_nf_nu_n(z)\sin\mu_nh-\\&-\sum_{\mu_n<\mu}f_n\int_z^{z+h}q(\xi)u_n(\xi)\cos\mu_n(\xi-z-h)\,d\xi+\\&+\sum_{\mu_n<\mu}f_n\int_{z-h}^{z}q(\xi)u_n(\xi)\cos\mu_n(z-\xi-h)\,d\xi.\end{aligned}$$

Analogous equality can be written for $\tilde{\sigma}'_\mu(z-h,f)$. Therefore, the following equality holds on $K_1$:

$$\begin{aligned}\sigma'_\mu(z-h,f)-\tilde{\sigma}'_\mu(z-h,f)=&\sum_{\mu_n<\mu}f_nu'_n(z+h)-\sum_{\tilde{\mu}_n<\mu}\tilde{f}_n\tilde{u}'_n(z+h)+\\&+\sum_{\mu_n<\mu}2\mu_nf_nu_n(z)\sin\mu_nh-\sum_{\tilde{\mu}_n<\mu}2\tilde{\mu}_n\tilde{f}_n\tilde{u}'_n(z)\sin\tilde{\mu}_nh-\\&-\sum_{\mu_n<\mu}f_n\int_z^{z+h}q(\xi)u_n(\xi)\cos\mu_n(\xi-x-h)\,d\xi+\\&+\sum_{\tilde{\mu}_n<\mu}\tilde{f}_n\int_z^{z+h}\tilde{q}(\xi)\tilde{u}_n(\xi)\cos\tilde{\mu}_n(\xi-z-h)\,d\xi+\\&+\sum_{\mu_n<\mu}f_n\int_{z-h}^{z}q(\xi)u_n(\xi)\cos\mu_n(z-\xi-h)\,d\xi-\\&-\sum_{\tilde{\mu}_n<\mu}\tilde{f}_n\int_{z-h}^{z}\tilde{q}(\xi)\tilde{u}_n(\xi)\cos\tilde{\mu}_n(z-\xi-h)\,d\xi.\end{aligned}$$

That is why we have the inequality

$$\|\sigma'_\mu(z-h,f)-\tilde{\sigma}'_\mu(z-h,f)\|^p_{L_p(K_1)}\leq C_p\|\sigma'_\mu(z+h,f)-\tilde{\sigma}'_\mu(z+h,f)\|^p_{L_p(K_1)}+$$

$$+ C_p \Big\| \sum_{\mu_n<\mu} 2\mu_n f_n u_n(z) \sin \mu_n h \Big\|^p_{L_p(K_1)} +$$
$$+ C_p \Big\| \sum_{\tilde{\mu}_n<\mu} 2\tilde{\mu}_n \tilde{f}_m \tilde{u}_n(z) \sin \tilde{\mu}_n h \Big\|^p_{L_p(K_1)} +$$
$$+ C_p \int_{K_1} \Big| \sum_{\mu_n<\mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - z - h)\, d\xi \Big|^p dz +$$
$$+ C_p \int_{K_1} \Big| \sum_{\tilde{\mu}_n<\mu} \tilde{f}_n \int_z^{z+h} \tilde{q}(\xi) \tilde{u}_n(\xi) \cos \tilde{\mu}_n(\xi - x - h)\, d\xi \Big|^p dz +$$
$$+ C_p \int_{K_1} \Big| \sum_{\mu_n<\mu} f_n \int_{z-h}^{z} q(\xi) u_n(\xi) \cos \mu_n(x - \xi - h)\, d\xi \Big|^p dz +$$
$$+ C_p \int_{K_1} \Big| \sum_{\tilde{\mu}_n<\mu} \tilde{f}_n \int_{z-h}^{z} \tilde{q}(\xi) \tilde{u}_n(\xi) \cos \tilde{\mu}_n(x - \xi - h)\, d\xi \Big| dz. \qquad (15)$$

Here and further, we denote by $C_p$ not necessarily equal positive constants.

In order to estimate the first member on the right-hand side of (15), we will use the estimate (5). Having in mind that $z + h \in K_2$ if $z \in K_1$, where $K_2 \overset{\text{def}}{=} [a + 2h, c + 2h] \subset G$, we have the inequalities

$$\|\sigma'_\mu(z + h, f) - \tilde{\sigma}'_\mu(z + h, f)\|^p_{L_p(K_1)} \le (c - a) C(K_2)^p \|f\|^p_{BV(G)}$$
$$\le C_p \|f\|_{BV(G)} \cdot \mu^{(1-1/p)p}. \qquad (16)$$

The next two members have the same "structure", and they will be estimated by the Riesz theorem. First we introduce a new function:

$$g(z) = \sum_{\mu_n<\mu} (2\mu_n f_n \sin \mu_n h) u_n(z), \qquad z \in G.$$

It belongs to $L_p(G) \subset L_2(G)$, and its Fourier coefficients (with respect to the system $\{u_n(z)\}_{n=1}^{\infty}$) are given by

$$g_n = \begin{cases} 2\mu_n f_n \sin \mu_n h & \text{if} \quad \mu_n < \mu, \\ 0 & \text{if} \quad \mu_n \ge \mu. \end{cases}$$

Let $r \in (1, 2]$ be a number such that $1/p + 1/r = 1$. By estimates (7) and (10), we obtain

$$\Big(\sum_{n=1}^{\infty} |g_n|^r\Big)^{1/r} \le C \|f\|_{BV(G)} \Big(\sum_{\mu_n<\mu} 1\Big)^{1/r}$$
$$\le C \|f\|_{BV(G)} \Big(\sum_{k=1}^{[\mu]} \Big(\sum_{k \le \mu_n < k+1} 1\Big)\Big)^{1/r} \le 2^{1/r} C A^{1/r} \|f\|_{BV(G)} \cdot \mu^{1/r}. \quad (17)$$

Hence, we can use the second part of the Riesz theorem: from estimate (12) it follows that the inequalities

$$\|g\|_{L_p(K_1)} \le \|g\|_{L_p(G)} \le (C(G))^{2/r-1} \Big(\sum_{n=1}^{\infty} |g_n|^r\Big)^{1/r}$$

are valid. That is why we can conclude, by (17), that for the second member it holds:

$$\Big\| \sum_{\mu_n<\mu} 2\mu_n f_n u_n(z) \sin \mu_n h \Big\|_{L_p(K_1)}^p \le C_p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p}. \tag{18}$$

The same estimate holds for the third member:

$$\Big\| \sum_{\tilde\mu_n<\mu} 2\tilde\mu_n \tilde f_n \tilde u_n(z) \sin \tilde\mu_n h \Big\|_{L_p(K_1)}^p \le \tilde C_p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p}. \tag{19}$$

In the case of the fourth member, using estimates (7)-(8), (10), and the Hölder inequality, we obtain

$$\int_{K_1} \Big| \sum_{\mu_n<\mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h)\, d\xi \Big|^p dz \le$$

$$\int_{K_1} \Big( \sum_{\mu_n<\mu} |\mu_n f_n| \Big| \frac{1}{\mu_n} \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h)\, d\xi \Big| \Big)^p \le$$

$$\Big( \sum_{\mu_n<\mu} |\mu_n f_n|^r \Big)^{p/r} \int_{K_1} \Big( \sum_{\mu_n<\mu} \Big| \frac{1}{\mu_n} \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h)\, d\xi \Big|^p \Big) dz \le$$

$$C_p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p} \|q\|_{L_1(G)}^p (c-a) \Big( \sum_{\mu_n<\mu} \frac{1}{\mu_n^p} \Big) \le C_p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p}.$$

Here $1/p + 1/r = 1$. Also we have in mind that

$$\sum_{\mu_n<\mu} \frac{1}{\mu_n^p} \le \sum_{k=1}^{\infty} \Big( \sum_{k\le\mu_n<k+1} \frac{1}{\mu_n^p} \Big) \le A \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

Therefore, the following estimate holds:

$$\int_{K_1} \Big| \sum_{\mu_n<\mu} f_n \int_z^{z+h} q(\xi) u_n(\xi) \cos \mu_n(\xi - x - h)\, d\xi \Big|^p dz \le$$

$$\le C_p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p}. \tag{20}$$

The estimates of the same form, with possibly different constants $C_p$, are valid for the last three members on the right-hand side of (15). So we get, by (14)–(16) and (18)–(20), the final estimate

$$\|\sigma'_\mu(x,f) - \tilde\sigma'_\mu(x,f)\|_{L_p((a,c))}^p \le C_p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p}. \tag{21}$$

Using the analogous argument, one can prove the estimate

$$\|\sigma'_\mu(x,f) - \tilde\sigma'_\mu(x,f)\|_{L_p((d,b))}^p \le C_p \|f\|_{BV(G)} \cdot \mu^{(1-1/p)p}. \tag{22}$$

Finally, by estimate (5), we obtain

$$\|\sigma'_\mu(x,f) - \tilde\sigma'_\mu(x,f)\|_{L_p(K)}^p \le (b-a) C(K)^p \|f\|_{BV(G)}^p \cdot \mu^{(1-1/p)p}. \tag{23}$$

Now, the estimate (6) follows from (13) and (21)–(23). The theorem is proved. ■

REFERENCES


[1] M. A. Naímark, *Lineínye differentsial'nye operatory*, Nauka, Moskva, 1969.

[2] A. M. Minkin, *Equiconvergence theorems for differential operators*, Journal of Math. Sci., **96**, 6 (1999), 3631–3715.

[3] V. A. Il'in, I. Ío, *Otsenka raznosti chastichnykh summ razlozhenií, otvechayushchikh dvum proizvol'nym neotritsatel'nym samosopyazhennym rasshireniyam dvukh operatorov tipa Shturma-Liuvillya, dlya absolyutno nepreryvnoí funktsii*, Diff. uravneniya **15**, 7 (1979), 1175–1193.

[4] I. S. Lomov, *Ob approksimatsii funktsií na otrezke spektral'nymi razlozheniyami operatora Shredingera*, Vestn. Mosk. un-ta., Ser.1, mat., meh., **4** (1995), 43–54.

[5] I. S. Lomov, *O skorosti ravnoskhodimosti ryadov Fur'e po sobstvennym funktsiyam operatorov Shturma-Liuvillya v integral'noí metrike*, Diff. uravneniya **18**, 9 (1982), 1480–1493.

[6] I. Ío, N. Lazhetich, *Otsenka raznosti proizvodnykh chastichnykh summ razlozhenií, otvechayushchikh dvum proizvol'nym neotritsatel'nym samosopyazhennym rasshireniyam dvukh operatorov tipa Shturma-Liuvillya, dlya absolyutno nepreryvnoí funktsii*, Diff. uravneniya **16**, 4 (1980), 598–619.

[7] N.L. Lažetić, O. Djordjević, *A local uniform estimate for the difference of derivatives of spectral expansions corresponding to self-adjoint one-dimensional Schrödinger operators*, submitted.

[8] V. A. Il'in, I. Ío, *Ravnomernaya otsenka sobstvennykh funktsií i otsenka sverkhu chisla sobstvennykh znachenií operatora Shturma-Liuvillya s potentsialom iz klassa $L^p$*, Diff. uravneniya **15**, 7 (1979), 1165–1174.

[9] L. V. Kritskov, *Ravnomernayua otsenka poryadka prisoedinennykh funktsií i raspredelenie sobbstvennykh znachenií odnomernogo operatora Shredingera*, Diff. uravneniya **25**, 7 (1989), 1121–1129.

[10] N. Lazhetich, *Ravnomernye otsenki dlya proizvodnyh sobstvennyh funktsií samosopryazhennogo operatora Shturma-Liuvillya*, Diff. uravneniya **17**, 11 (1981), 1978–1983.

[11] A. Zygmund, *Trigonometric Series*, Volume I and II, Cambridge, 1968.


(received 27.12.2002)

Faculty of Mathematics, Studentski trg 16, P.O. Box 550, 11000 Beograd, Yugoslavia

E-*mail*: lazetic@matf.bg.ac.yu

Faculty of Ogranizational Sciences, Jove Ilića 154, 11000 Beograd, Yugoslavia

E-*mail*: oliveradj@fon.bg.ac.yu