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Empirical (α, β)-acceptable optimal values to full fuzzy linear fractional programming problems

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Abstract

In this paper we aim to provide empirical solutions to a special class of full fuzzy linear fractional programming problems. We use trapezoidal fuzzy numbers to describe the parameters and derive empirical shape of the membership of the goal function optimal values of the problem. Our approach essentially follows the extension principle, and is based on solving crisp quadratic optimization problems. The model we propose treats in different ways, through two independent parameters, the objective function coefficients and coefficients in the constraints. To illustrate our theory, we solve a relevant instance and compare our numerical results with the numerical results recalled from the recent literature.

Keywords: Full fuzzy linear fractional programming; Trapezoidal fuzzy numbers; Extension principle.
2000 MSC: 90C32, 90C70

1. Introduction

The concept of fuzzy set was introduced by Zadeh [1] as a tool able to handle mathematically the uncertainty. Zimmerman [2] was the first one to use the fuzzy concepts in mathematical programming proposing a fuzzy approach to linear programming with multiple objective function. Later on, Zimmermann [3] discussed the applications of fuzzy set theory to mathematical programming in a wider sense. More recent, Zadeh [4] readdress the fuzzy concepts discussing about advantages of using fuzzy logic in solving real life problems.

Several papers that address full fuzzy programming problems can be found in the recent literature (see for instance [5], [6], [7], [8], [9], [10], [11]). We briefly present two of them that are relevant to our current work.

Chinnadurai and Muthukumar [5] used the concept of (α, r)-acceptable optimal value by treating distinctly the desired level associated to the objective values and the acceptable level for the constraints. They applied it to a linear fractional programming problem with fuzzy coefficients and fuzzy decision variables. They obtained an (α, r)-acceptable optimal value to such problems by solving a certain crisp bi-objective linear fractional programming problem. They constructed the membership function of the optimal values numerically, solving a crisp problem for several fixed values of α and r.

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Ebrahimnejad et al. [6] overcame a shortcoming that had arisen in [5]; added a new constraint to the mathematical model to ensure the non-negativity of the fuzzy optimal solution; and then extended the solution approach to solving full fuzzy linear fractional programming problems with trapezoidal fuzzy parameters. We compare our solution approach that solves the same class of problems to their approach through a numerical example, emphasizing the advantages of the novel approach.

Empirical versus analytical solutions to linear programming problems with fuzzy parameters and full fuzzy linear programming problems were analyzed in [12]. We extend their results to a certain class of full fuzzy fractional programming problems.

The approach we propose strictly follows the extension principle and incorporates the aggregation of fuzzy quantities in the optimization step. An incipient but solid mathematical foundation for the extension principle-based optimization can be found in Diniz et al. [13]. They discussed the optimization of a fuzzy-valued function as a process that must comply to Zadeh's extension principle [1]. Their objective function was a Zadeh's extension of a function with respect to a parameter and an independent variable; but no constraints were included. Also, Kupka [14] recently introduced some results on how to approximate a Zadeh's extension of a given function, and studied how the quality of the approximation varies with respect to the choice of the metric on the universe of the fuzzy sets.

We organize the rest of the paper as follows: Section 2 introduces the basic notation and terminology concerning the fuzzy sets theory, and fractional programming in fuzzy environment that will be used in the sequel; Section 3 briefly presents the theoretical foundations for our solution approach; Section 4 is included to illustrate our theoretical statements, and reports our numerical results compared to results from the recent literature. The final conclusion and some directions for further research are included in Section 5.

2. Preliminaries

In this section we briefly present the notation and terminology related to fuzzy sets theory, and fractional programming that will be useful in the sequel.

A fuzzy set $\widetilde{A}$ of the universe X is by definition a set of pairs $(x, \mu_{\widetilde{A}}(x))$ where the first component is an element of the universe X and the second component is the membership degree with which the first component belongs to the fuzzy set. The function $\mu_{\widetilde{A}}$ is called the membership function of the fuzzy set, and takes values in $[0, 1]$.

A trapezoidal fuzzy number is an especial fuzzy set of the universe $\mathbb{R}$, the set of real numbers. Each trapezoidal fuzzy number can be described by a quadruple (a_1, a_2, a_3, a_4). The α-cut of a trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is the interval $\left[a^L(\alpha), a^U(\alpha)\right]$, where $a^L(\alpha) = \alpha a_2 + (1-\alpha) a_1$ and $a^U(\alpha) = \alpha a_3 + (1-\alpha) a_4$. More details about fuzzy numbers can be found in (see Zadeh [1] for additional details).

Zadeh [15] introduced the extension principle to extend the classic arithmetic to a realistic arithmetic over the set of fuzzy numbers. The extension principle defines the fuzzy set $\widetilde{B}$ with the help of the membership function

$$\mu_{\widetilde{B}}(y) = \begin{cases} \sup\limits_{(x_1,\dots,x_p)\in f^{-1}(y)} \left(\min\left\{\mu_{\widetilde{A}_1}(x_1), \dots, \mu_{\widetilde{A}_r}\left(x_p\right)\right\}\right), & f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

where $\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_p$ are fuzzy sets over their universes $X_1, X_2, \dots, X_p$; the set $Y = X_1 \times ... \times X_p$ is the universe of $\widetilde{B}$, and function f is a classic function that maps $X_1 \times ... \times X_p$ to Y. More details about fuzzy numbers, their arithmetic, and applications to mathematical programming can be found in Zimmerman [3].

The full fuzzy linear fractional programming problem is analogue to the linear fractional programming problem, and involves fuzzy numbers on the positions of both coefficients and decision variables. The general mathematical model of a full fuzzy linear fractional programming problem is given in (1).

$$\begin{array}{ll} \max & \left(\widetilde{c}^T \widetilde{x} + \widetilde{c}_0\right) / \left(\widetilde{d}^T \widetilde{x} + \widetilde{d}_0\right) \\ \text{s.t.} & \\ & \widetilde{A}\widetilde{x} \leq \widetilde{b}, \\ & \widetilde{x} \geq 0. \end{array} \tag{1}$$

In what follows we discuss the concept of (α,β)-acceptable optimal values applied to Problem (1) by using the α-cuts with two different parameters: α for describing the objective function coefficients, and β for describing the constraints coefficients.

Generally, whenever a fuzzy optimization problem is solved it has to be clearly specified how the fuzzy quantities are aggregated and compared as well as how the optimization is performed. As it will be seen in the next section, we choose to globally employ the extension principle, thus succeeding to unify the fuzzy quantities aggregation, comparison and optimization in a single stage when solving Problem (1).

3. Our theoretical results

To follow the extension principle and the methodology presented by Stanojević and Stanojević [12], we make use of the crisp linear fractional programming problem

$$\begin{array}{ll} \max & \left(c^T x + c_0\right) / \left(d^T x + d_0\right) \\ \text{s.t.} & \\ & Ax \leq b, \\ & x \geq 0. \end{array} \tag{2}$$

The Monte Carlo simulation-based algorithm that we propose consists in solving Problem (2) with randomly generated coefficients A, b, c, c_0, d and d_0 between their lower and upper bounds. For fixed values of $\alpha,\beta \in [0,1]$, the coefficients bounds are provided by the α- and β-cuts of their corresponding fuzzy numbers, namely

$$\begin{array}{lll} c_j^L(\alpha) \leq c_j \leq c_j^U(\alpha), & j = 0,\ldots,n, & \\ d_j^L(\alpha) \leq d_j \leq d_j^U(\alpha), & j = 0,\ldots,n, & \\ a_{ij}^L(\beta) \leq a_{ij} \leq a_{ij}^U(\beta), & i = 1,\ldots,m, & j = 1,\ldots,n, \\ b_i^L(\beta) \leq b_i \leq b_i^U(\beta), & i = 1,\ldots,m, & \end{array} \tag{3}$$

where a_{ij} represent the components of the constraints matrix A; c_j and d_j are the objective function coefficients; and b_i is the right-hand side of the i-th constraint. After each random generation followed by optimization, the optimal value of the objective function is memorized together with its (α,β)-membership degrees computed as

$$\begin{array}{l} \alpha = \min\left(\left\{\mu_{\widetilde{c}_j}\left(c_j\right) | j = \overline{0,n}\right\} \cup \left\{\mu_{\widetilde{d}_j}\left(d_j\right) | j = \overline{0,n}\right\}\right) \\ \beta = \min\left(\left\{\mu_{\widetilde{a}_{ij}}\left(a_{ij}\right) | i = \overline{1,m}, j = \overline{1,n}\right\} \cup \left\{\mu_{\widetilde{b}_i}\left(b_i\right) | i = \overline{1,m}\right\}\right) \end{array} \tag{4}$$

The idea is formalized in Algorithm 1.

Algorithm 1

Input: A natural number $q \in N$; a sequence $\alpha_1, \alpha_2, \ldots, \alpha_q \in [0,1]$; a value $\beta \in [0,1]$; the membership functions of the coefficients $\widetilde{A}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{c}_0$, $\widetilde{d}$ and $\widetilde{d}_0$ of Problem (1).

1: Set $L = \emptyset$.
2: **for** $k = \overline{1,q}$ **do**
3: Randomly generate c_0, d_0, c_j, d_j, a_{ij}, b_i, $j = \overline{1,n}$, $i = \overline{1,m}$ within their bounds given in (3) for α_k and β.
4: Compute α using (4).
5: Solve Problem (2) and derive the optimal value z^*.
6: Set $L = L \cup \{(z^*, \alpha, \beta)\}$
7: **end for**

Output: The list L, containing the optimal values paired with their corresponding (α,β)-acceptability degree.

The envelopes of the generated maximal values of the objective function can be obtained by solving the nonlinear Model (5) over the feasible set of the variables x_j, c_j, d_j, a_{ij} and b_i; and for fixed values of the parameters α and β.

$$
\begin{array}{lll}
\max & \left(\sum_{j=1}^{n} c_j x_j + c_0\right) / \left(\sum_{j=1}^{n} d_j x_j + d_0\right) & \\
\text{s.t.} & & \\
& \sum_{j=1}^{n} a_{ij} x_j \le b_i, & i = 1, \dots, m, \\
& x_j \ge 0. & j = 1, \dots, n, \\
& c_j^L(\alpha) \le c_j \le c_j^U(\alpha), & j = 0, \dots, n, \\
& d_j^L(\alpha) \le d_j \le d_j^U(\alpha), & j = 0, \dots, n, \\
& a_{ij}^L(\beta) \le a_{ij} \le a_{ij}^U(\beta), & i = 1, \dots, m, \quad j = 1, \dots, n, \\
& b_i^L(\alpha) \le b_i \le b_i^U(\alpha), & i = 1, \dots, m.
\end{array} \tag{5}
$$

4. Our numerical results

To illustrate our theoretical results we report our numerical results obtained by solving a full fuzzy linear fractional programming problem recalled from [5] and [6]. The trapezoidal fuzzy numbers that are coefficients of the fractional programming problem are given in Table 1.

Table 1. The trapezoidal fuzzy numbers representing the coefficients of Problem (1) solved in our numerical example

The coefficients of the obective function			The oefficients of the constraints		
$c_1 = (3, 5, 8, 13)$	$c_2 = (2, 3, 5, 7)$	$c_0 = (0, 0, 0, 0)$	$a_{11} = (0, 1, 2, 3)$	$a_{12} = (1, 2, 3, 4)$	$b_1 = (3, 5, 8, 14)$
$d_1 = (3, 5, 8, 13)$	$d_2 = (1, 2, 4, 6)$	$d_0 = (0, 1, 2, 3)$	$a_{21} = (1, 2, 3, 4)$	$a_{22} = (0, 1, 1, 2)$	$b_2 = (3, 4, 5, 8)$

Our final numerical results compared to the results generated by following a methodology from the literature (i.e. proposed in [6]) are shown in Table 2, and graphed in Figures 1 and 2.

Table 2. The comparative results of the maximal objective values derived by our approach and the approach introduced in [6]

β	z_3^*, [6]	z_3^* (ext. princ.)	z_4^*, [6]	z_4^* (ext. princ.)
1.00	1.041	2.241	1.451	7.00
0.70	2.175	2.320	92232.068	7.00
0.50	2.193	2.360	55691.135	7.00
0.25	2.210	2.401	37086.417	7.00
0.00	2.224	2.434	27564.008	7.00

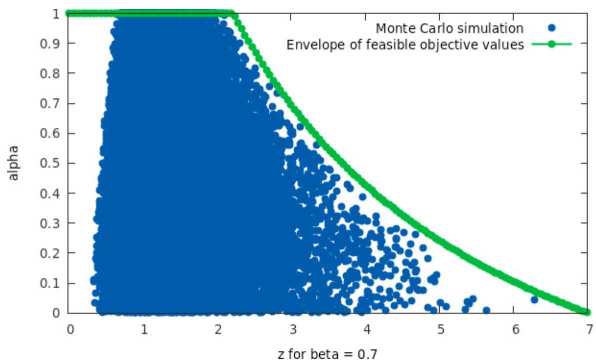

Fig. 1. The generated points obtained by running the Monte Carlo simulation-based algorithm, and the approximate envelope of the maximal objective values derived by solving Problem (5)

Before analyzing our results we have to emphasize that treating the decision variables as triangular or trapezoidal fuzzy numbers, and applying algebraic operators on their components (i.e. exclusively on their values

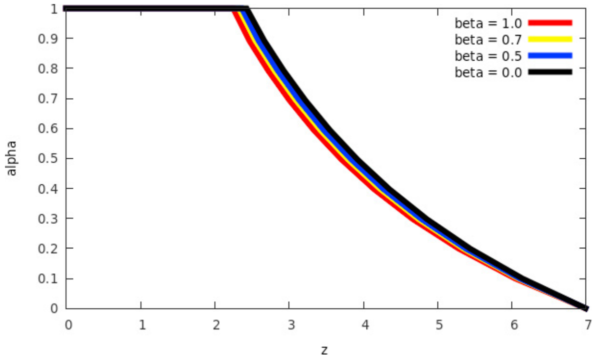

Fig. 2. The maximal values of the objective function with respect to α, for various values of β obtained by solving Problem (5)

with minimal non-zero and maximal amplitude) before optimization – as it was done in [5] and [6] – waste the relevance of the derived fuzzy set solution to the original problem, since the essence of the aggregation of fuzzy quantities aiming to assure the best values for the objective function is lost.

Let $\left(z_1^*, z_2^*, z_3^*, z_4^*\right)$ be the trapezoidal fuzzy number representing the approximation of the fuzzy set of the maximal values of the objective function. For $\beta \in \{0, 0.25, 0.5, 0.7, 1\}$, Table 2 reports the values z_3^* and z_4^* that are relevant when comparing the two approaches. For any fixed value of β the value z_3^* represents the maximal value of the objective function obtained for $\alpha = 1$, while the value z_4^* represents the maximal value of the objective function obtained for $\alpha = 0$.

Table 2 brings out that for $\alpha = 1$, and all tested values of β the maximal value of the objective function seen on our envelope has the same order of magnitude like the one obtained by the approach from the literature. On the other side, for $\alpha = 0$, the maximal values obtained in [6] are inconsistent. From these results we conclude that the approach from the literature are in fact misleading, relying on an artificial aggregation of fuzzy quantities followed by a classic optimization.

The maximal values of the objective function for different combinations of the coefficients, obtained by Monte Carlo simulation are presented in Figure 1. Parameter β was set to 0.7 for all results presented in Figure 1. In the same figure, we also present the envelope of all feasible values of the objective function. It can be seen how the right side of the envelope follows the empirical shape obtained by the Monte Carlo simulation. Combining information from Figure 1 with the results reported in Table 2 we conclude that Ebrahimnejad et al.'s approach [6] does not work in accordance to the extension principle.

For any fixed value of α, Figure 2 shows that very small differences appear between the maximal values of the objective function when the parameter β is varying.

Our goal for running these experiments was two-fold: (i) to test whether the approach from the literature derives solutions in accordance to the extension principle or not; and (ii) to derive relevant information about solutions that comply to the extension principle.

5. Conclusions

The main contribution of this paper is that it provides a simple methodology to validate approaches to solving full fuzzy linear fractional programming problems via a Monte Carlo simulation algorithm. In addition, we empirically derived and approximated the shapes of the membership function of the fuzzy set of the optimal objective values.

Our approach strictly followed the extension principle, and crisp quadratic optimization problems were effectively solved. We treated in different ways, through two independent parameters, the objective function coefficients on one side and the coefficients in the constraints on the other side. The formalization was used before in the literature but the way we disolved the fuzziness has brought the essential distinction between our results and the results from the literature.

Whenever a solution approach to fuzzy optimization problems strictly follows the extension principle the ranking of the involved fuzzy quantities is avoided. This fact is a real advantage, since there are many ranking functions defined in the literature, non of them can be successfully applied to all classes of optimization problems, each of them might generate a solution approach to certain classes of problems, and any comparison of their effectiveness is almost impossible.

To illustrate our theory, we solved a relevant instance and compared our numerical results with the numerical results found in the recent literature.

In our future research we intend to develop a methodology to derive the lower bound of the maximal objective values, and focus on applying the Monte Carlo simulation method to a wider class of optimization problems under uncertainty aiming to identify existing methods in the literature that should be adjust to follow the extension principle.

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