Applications of Fuzzy Systems in Mechanical Systems - Research Article

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Computationally semi-numerical technique for solving system of intuitionistic fuzzy differential equations with engineering applications

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Abstract

Some complex problems in science and engineering are modeled using fuzzy differential equations. Many fuzzy differential equations cannot be solved by using exact techniques because of the complexity of the problems mentioned. We utilize analytical techniques to solve a system of fuzzy differential equations because they are simple to use and frequently result in closed-form solutions. The Generalized Modified Adomian Decomposition Method is developed in this article to compute the analytical solution to the linear system of intuitionistic triangular fuzzy initial value problems. The starting values in this case are thought of as intuitionistic triangular fuzzy numbers. Engineering examples, such as the Brine Tanks Problem, are used to demonstrate the proposed approach and show how the series solution converges to the exact solution in closed form or in series. The corresponding graphs at different levels of uncertainty show the example's numerical outcomes. The graphical representations further demonstrate the effectiveness and accuracy of the proposed method in comparison to Taylor's approaches and the classical Decomposition method.

Keywords

System of fuzzy differential equation, analytical technique, triangular intuitionistic fuzzy number, fuzzy set, engineering applications

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Introduction

System of differential equations are used to model physical and technical problems in a wide range of disciplines, including visco elasticity, biology, physics, solid and fluid mechanics, and many more. In general, the parameters, variables and initial conditions within a model are considered as being defined exactly. In reality there are only vague, imprecise, or incomplete information about the variables and parameters available. This can result from errors in measurement, observation, or experimental data; application of different operating conditions; or maintenance induced errors. General

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differential equations or system of differential equations can be transformed into fuzzy differential or system of differential equations by employing a fuzzy environment for the parameters, variables, and initial conditions instead of crisp ones in order to overcome these uncertainties or lack of precision. Due to the limitations of fuzzy arithmetic, system of fuzzy differential equations may be difficult to solve precisely in practical applications, requiring the employment of reliable and effective semi-numerical techniques.

The concept of fuzzy set theory was firstly introduced by Zadeh[1] in 1965, as the extension of classical set theory. The concept of fuzzy set theory has been applied to various fields of science and engineering to handle vagueness and uncertainty. In 1987, Kandel and Byatt[2] introduced the fuzzy differential equations. The fuzzy differential equations have been applied in numerous daily life problems.[3,4] Vasavi et al.[5] discussed fuzzy differential for cooling problems. Sindu Devi and Ganesan[6] used fuzzy differential equations in modeling electric circuit problems. Ahmad et al.[7] studied a mathematical method to find the solution of fuzzy integro differential equations. Sadeghi et al.[8] studied the system of fuzzy differential equations. Buckley et al.[9] find the solution of system of first order linear fuzzy differential equations by extension principle. Fard[10] introduced numerical technique for computing SFDE. Hashemi et al.[11] finds the series solution of SFDE. In 1986, Atanassov[12] introduced an extension of fuzzy set theory known as intuitionistic fuzzy set. The intuitionistic fuzzy set[13] not only provides the information about membership values but also the non-membership values respectively, and so that the sum of both values is less than 1. Intuitionistic fuzzy differential equations are being studied widely and being used in various fields of Physics, Chemistry, Biology as well as among other fields of science and engineering. Melliani and Chadli[14] obtained the approximate and numerical solutions of intuitionistic fuzzy differential equations with linear differential operators. Akin and Bayeğ[15] studied a method to find general solution of second order intuitionistic fuzzy differential equation and to solve the system of intuitionistic fuzzy differential equations with intuitionistic fuzzy initial values. Prasad Mondal and Kumar Roy[16] studied the generalized intuitionistic fuzzy Laplace transform method and to solve the system of differential equations with initial value as triangular intuitionistic fuzzy number. Saw et al. introduced a method for solving system of linear intuitionistic fuzzy equations.[17–20]

Adomian[21,22] first introduced the semi-analytical Adomian and Rach Decomposition Method (ADM) in the 1980s. Differential equations, algebraic equations, and integral equations can all be effectively solved using this technique. The Generalized Modified Adomian Decomposition Method (GMADM), which can be used to solve a system of linear intuitionistic fuzzy differential equations whose initial values are generalized trapezoidal intuitionistic fuzzy numbers, is introduced in this article. Wazwaz[23] proposed this modification. He suggested the ADM to make a substantial improvement. Wazwaz divides the original function into two components in this modification, allocating one to the series' first term and the other to its second. The series that is produced as a result of this alteration is distinct. The efficiency of this approach solely depends on the selection of the components into which the original function will be split.

First order system of fuzzy differential equations is important among all the fuzzy differential equations. There are many approaches to solve the SFDEs. Buckley et al.[9] solving the linear system of first order ordinary differential equations with fuzzy initial conditions by extension principle using triangular fuzzy number. The geometric approach is developed by Gasilova et al. [24] and series solution is developed by Hashemi et al.[11] Mondal and Roy[25] studied strong and weak solution of first order homogeneous intuitionistic fuzzy differential equation, subsequently, who studied system of differential equation in literature.[13] Melliani et al.[26] discussed the existence and uniqueness of the solution of the intuitionistic fuzzy differential equation and its system using the analytical technique. Therefore, finding an efficient and accurate algorithm for investigating FIE has been one the hot areas of research in recent time. To achieve these goals, various methods and procedures were used to handle integral equations, for details, see Sadeghi[8] and Nirmala and Pandian.[27]

Motivated by the aforesaid work, in this article, we use a GMADM to solve the fuzzy system of intuitionistic differential equations.

The main contributions of this research work are summarized below.

- GMADM is used to solve a system of differential equations using initial conditions as a triangular intuitionistic fuzzy number.
- In order to solve a system of fuzzy intuitionistic differential equations that have not before been explored, the computational complexity of the suggested GMADM is discussed.
- Applications of system of intuitionistic fuzzy differential equations in mechanical engineering are taken into consideration in a intuitionistic fuzzy environment.
- Computational tools are used to evaluate the effectiveness and applicability of the suggested method.

This article is organized as follows: We went over some important concepts in section 2 that will be used in the sections following. The proposal's Section 3 contains our suggested procedure. Applications in section 4 have demonstrated the efficiency of this method. Section 5, concludes the paper.

Preliminaries

In this section, the fundamental definitions of fuzzy set and intuitionistic fuzzy set[28] are presented.

Definition 2.1.[1] *If $\hat{S}$ is a collection of objects and its objects are denoted by $\hat{s}$, then fuzzy set F in $\hat{S}$ is a set of ordered pairs*

$$F = \{(\hat{s}, \mu_F(\hat{s})) \mid \hat{s} \in \hat{S}\},$$

where

$$\mu_F(\hat{s}) = \hat{S} \rightarrow [0, 1],$$

defines the grade of membership or simply called membership function.

Definition 2.2. *α_1 – cut of a fuzzy F se*[9] *is a crisp set F_{α_1} and is defined by*

$$F(\alpha_1) = \{\hat{s} \mid \mu_F(\hat{s}) \geqslant \alpha_1\},$$

where $F = \{(\hat{s}, \mu_F(\hat{s}))\}$ and $\alpha \epsilon [0, 1]$.

Definition 2.3. *A fuzzy set F is said to be convex fuzzy set*[3] *if*

$$\mu_F(\tau \hat{s}_1 + (1 - \tau)\hat{s}_2) \geqslant \min(\mu_F(\hat{s}_1), \mu_F(\hat{s}_2),$$

for all $\hat{s}_1, \hat{s}_2 \epsilon \hat{S}$ and $\tau \epsilon [0, 1]$.

Definition 2.4. *A fuzzy set F is said to be normal fuzzy set*[31] *if there exists an element $\hat{s} \epsilon \hat{S}$ such that*

$$\mu_F(\hat{s}) = 1.$$

Definition 2.5. *A fuzzy number*[9] *$\overset{*}{F}$ is a fuzzy set on a real line if it is normal as well as a convex fuzzy set.*

Intuitionistic fuzzy set

Definition 2.6. *Let $\hat{S}$ be the largest set under consideration then an intuitionistic fuzzy I set*[28] *in $\hat{S}$ is defined as:*

$$I = \{(\hat{s}, \mu_I(\hat{s}), \nu_I(\hat{s})) : \hat{s} \epsilon \hat{S}\},$$

where

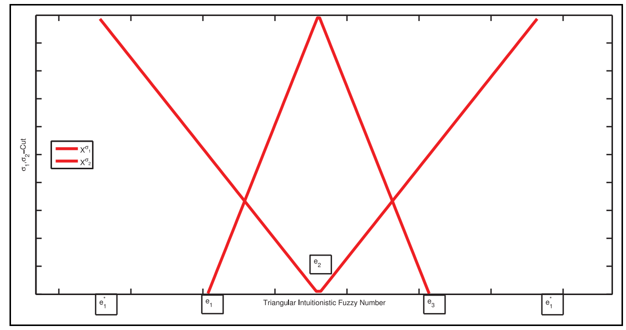

Figure 1. Triangular intuitionistic fuzzy number.

$$\mu_I(\hat{s}) = \hat{S} \rightarrow [0, 1],$$
$$\nu_I(\hat{s}) = \hat{S} \rightarrow [0, 1].$$

defines the grade of membership and grade of non-membership respectively, of the element $\hat{s} \epsilon \hat{S}$ to the set $I \subseteq \hat{S}$ and for every element $\hat{s} \epsilon \hat{S}$, $0 < \mu_I(\hat{s}) + \nu_I(\hat{s}) \leqslant 1$.

Definition 2.7. (α_1, α_2) *-cut of an intuitionistic fuzzy set I is crisp set $I_{\alpha\beta}$ which is defined as*[29]:

$$I_{\alpha\beta} = \{\hat{s} \mid \mu_I(\hat{s}) \geqslant \alpha_1 \nu_I(\hat{s}) \leqslant \alpha_2\},$$

where $I = \{(\hat{s}, \mu_I(\hat{s}), \nu_I(\hat{s})) : \hat{s} \epsilon \hat{S}\}, \alpha_1, \beta \epsilon [0, 1]$ and $\alpha_1 + \alpha_2 \leqslant 1$.

Definition 2.8. *An intuitionistic fuzzy set I is said to be convex for membership function*[30] *if*

$$\mu_I(\tau \hat{s}_1 + (1 - \tau)\hat{s}_2) \geqslant \min(\mu_I(\hat{s}_1), \mu_I(\hat{s}_2),$$

for all $\hat{s}_1, \hat{s}_2 \epsilon \hat{S}$ and $\tau[0, 1]$.

Triangular intuitionistic fuzzy number (TIFNs) is geometrically presented in Figure 1.

Figure 1: Shows membership and non-membership function of triangular intuitionistic fuzzy number.

Definition 2.9. *An intuitionistic fuzzy set I is said to be concave for non-membership function*[13] *if*

$$\nu_I(\tau \hat{s}_1 + (1 - \tau)\hat{s}_2) \leqslant \max(\nu_I(\hat{s}_1), \nu_I(\hat{s}_2),$$

for all $\hat{s}_1, \hat{s}_2 \epsilon \hat{S}$ and $\tau \epsilon [0, 1]$.

Definition 2.10.[30] *An intuitionistic fuzzy set I is said to be normal if there exist an $\hat{s} \epsilon \hat{S}$ such that*

$$\mu_I(\hat{s}) = 1 \text{so} \nu_I(\hat{s}) = 0.$$

Definition 2.11. *An intuitionistic fuzzy number* $\overset{\blacklozenge}{N}$ *is an intuitionistic fuzzy subset on a real line that is:*[30]

- Normal intuitionistic fuzzy set.
- Convex intuitionistic fuzzy set for membership function.
- Concave intuitionistic fuzzy set for non-membership function.

Definition 2.12. *An intuitionistic fuzzy number is said to be triangular intuitionistic fuzzy number*[13] $T=(r_1,r_2,r_3;u_1,u_2,u_3)$ *if its membership and non-membership function are defined as follows*:

$$\mu_T(\hat{s}) = \begin{cases} \dfrac{\hat{s}-r_1}{r_2-r_1}, r_1 \leqslant \hat{s} \leqslant r \\ \dfrac{r_3-\hat{s}}{r_3-r_2}, r_2 \leqslant \hat{s} \leqslant r_3 \\ 0, \text{otherwise} \end{cases} \quad \text{and}$$

$$\nu_T(\hat{s}) = \begin{cases} \dfrac{u_2-\hat{s}}{u_2-u1}, u_1 \leqslant \hat{s} \leqslant u_2 \\ \dfrac{\hat{s}-u_2}{u_3-u_2}, u_2 \leqslant \hat{s} \leqslant u_3 \\ 1, \text{otherwise.} \end{cases},$$

where $u_1 < r_1 \leqslant r_2 \leqslant r_3 \leqslant u_3$.

Arithmetic operations on TIFNs

Definition 2.13. *Let* $T_1=(r_1,r_2,r_3;u_1,u_2,u_3)$ *and*

$T_2=(v_1,v_2,v_3;w_1,w_2,w_3)$ be two TIFNs and ϕ be a real number.[30] Then

$$\left.\begin{aligned} T_1+T_2 &= \begin{pmatrix} r_1+v_1, r_2+v_2, \\ r_3+v_3; u_1+w_1, \\ u_2+w_2, u_3+w_3 \end{pmatrix}, \\ T_1-T_2 &= \begin{pmatrix} r_1-v_3, r_2-v_1, \\ r_3-v_1; u_1-w_3, \\ u_2-w_2, u_3-w_1 \end{pmatrix}, \\ T_1\times T_2 &= \begin{pmatrix} r_1v_1, \\ r_2v_2, \\ r_3v_3; \\ u_1w_1, \\ u_2w_2, \\ u_3w_3 \end{pmatrix}; T_1, T_2>0. \\ T_1 \div T_2 &= \begin{pmatrix} \frac{r_1}{v_3}, \frac{r_2}{v_2}, \frac{r_3}{v_1}; \\ \frac{u_1}{w_3}, \frac{u_2}{w_2}, \frac{u_3}{w_1} \end{pmatrix}; T_2>0. \\ \phi T_1 &= \begin{pmatrix} \phi t_1, \phi t_2, \phi t_3; \\ \phi u_1, \phi u_2, \phi u_3 \end{pmatrix}; \phi>0, \\ \phi T_1 &= \begin{pmatrix} \phi t_3, \phi t_2, \phi t_1; \\ \phi u_3, \phi u_2, \phi u_1 \end{pmatrix}; \phi<0. \end{aligned}\right\} \tag{1}$$

The generalized modified Adomian decomposition method

A key concept is that the Adomian decomposition series expansion about the initial solution component function that permits solution by recursion, in which the aforesaid rearrangement is accomplished through the choice of the recursion scheme. The modified ADM yields a rapidly convergent sequence of analytic functions as the approximate solutions of the original mathematical model. Thus the Modified Adomian decomposition method subsumes even the classic power series method while extending the class of amenable non-linearity to include any analytic non-linearity. Here we generalized the MADM to GMADM to solve system of intuitionistic triangular fuzzy differential equation. Let us consider the system of intuitionistic fuzzy differential equations with linear differential operator as follows:

$$\left.\begin{aligned} &\pounds x(r)+Rx(r)+Ry(r) \\ &+N(r,x(r),y(r))=g(r), \\ &\pounds y(r)+Rx(r)+Ry(r)+ \\ &N(r,x(r),y(r))=h(r). \end{aligned}\right\} \tag{2}$$

where $\pounds$ is the highest order linear differential operator, R is the remaining part of the linear differential operator, N may be linear or nonlinear function of r, $x(r)$ and $y(r)$, $g(r)$, and $h(r)$ are inhomogenous terms. Here, in this case we take N as a linear function of $x(r), y(r)$ and r.Taking $(\vartheta_1,\vartheta_2)$-cut of (2), we get;

$$\left.\begin{aligned} &\pounds([x_1(\sigma_1^*),x_2(\sigma_1^*)];[x_1(\sigma_2^*),x_2(\sigma_2^*)]) \\ &+R([x_1(\sigma_1^*),x_2(\sigma_1^*)]; \\ &x_1(\sigma_2^*),x_2(\sigma_2^*)]) \\ &+R([y_1(\sigma_1^*),y_2(\sigma_1^*)];[y_1(\sigma_2^*),y_2(\sigma_2^*)]) \\ &+([N_1(r,x_1(\sigma_1^*),y_1(\sigma_1^*)), \\ &N_2(r,x_2(\sigma_1^*),y_2(\sigma_1^*))]; \\ &[N_1(r,x_1(\sigma_2^*),y_1(\sigma_2^*)), \\ &N_2(r,x_2(\sigma_2^*),y_2(\sigma_2^*))]) \\ &=([g_1(\sigma_1^*),g_2(\sigma_1^*)];[g_1(\sigma_2^*),g_2(\sigma_2^*)]), \\ &\pounds([y_1(\sigma_1^*),y_2(\sigma_1^*)];[y_1(\sigma_2^*),y_2(\sigma_2^*)])+ \\ &R([x_1(\sigma_1^*),x_2(\sigma_1^*)]; \\ &[x_1(\sigma_2^*),x_2(\sigma_2^*)])+ \\ &R([y_1(\sigma_1^*),y_2(\sigma_1^*)];[y_1(\sigma_2^*),y_2(\sigma_2^*)]) \\ &+([N_1(r,x_1(\sigma_1^*),y_1(\sigma_1^*)), \\ &N_2(r,x_2(\sigma_1^*),y_2(\sigma_1^*))]; \\ &[N_1(r,x_1(\sigma_2^*),y_1(\sigma_2^*)), \\ &N_2(r,x_2(\sigma_2^*),y_2(\sigma_2^*))]) \\ &=([h_1(\sigma_1^*),h_2(\sigma_1^*)];[h_1(\sigma_2^*),h_2(\sigma_2^*)]), \end{aligned}\right\} \tag{3}$$

where $\sigma_1^*=(r,\vartheta_1)$ and $\sigma_2^*=(r,\vartheta_2)$ From (3), we obtain the following equations:

$$\left.\begin{aligned}&£x_1(\sigma_1^*) + Rx_1(\sigma_1^*) + Ry_1(\sigma_1^*) + \\ &N_1(r, x_1(\sigma_1^*), y_1(\sigma_1^*)) = g_1(\sigma_1^*), \\ &£y_1(\sigma_1^*) + Rx_1(\sigma_1^*) + Ry_1(\sigma_1^*) + \\ &N_1(r, x_1(\sigma_1^*), y_1(\sigma_1^*)) = h_1(\sigma_1^*).\end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned}&£x_2(\sigma_1^*) + Rx_2(\sigma_1^*) + Ry_2(\sigma_1^*) + \\ &N_2(r, x_2(\sigma_1^*), y_2(\sigma_1^*)) = g_2(\sigma_1^*), \\ &£y_2(\sigma_1^*) + Rx_2(\sigma_1^*) + Ry_2(\sigma_1^*) + \\ &N_2(r, x_2(\sigma_1^*), y_2(\sigma_1^*)) = h_2(\sigma_1^*).\end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned}&£x_1(\sigma_2^*) + Rx_1(\sigma_2^*) + Ry_1(\sigma_2^*) + \\ &N_1(r, x_1(\sigma_2^*), y_1(\sigma_2^*)) = g_1(\sigma_2^*), \\ &£y_1(\sigma_2^*) + Rx_1(\sigma_2^*) + Ry_1(\sigma_2^*) + \\ &N_1(r, x_1(\sigma_2^*), y_1(\sigma_2^*)) = h_1(\sigma_2^*).\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}&£x_2(\sigma_2^*) + Rx_2(\sigma_2^*) + Ry_2(\sigma_2^*) + \\ &N_1(r, x_2(\sigma_2^*), y_2(\sigma_2^*)) = g_2(\sigma_2^*), \\ &£y_2(\sigma_2^*) + Rx_2(\sigma_2^*) + Ry_2(\sigma_2^*) + \\ &N_1(r, x_2(\sigma_2^*), y_2(\sigma_2^*)) = h_2(\sigma_2^*),\end{aligned}\right\} \quad (7)$$

Applying the $£^{-1}$ operator on both sides of (4)−(7), we get;

$$\left.\begin{aligned}x_1(\sigma_1^*) &= \Psi_1(\sigma_1^*) - £^{-1}(Rx_1(\sigma_1^*)) \\ &-£^{-1}(Ry_1(\sigma_1^*)) \\ &-£^{-1}(N_1(r, x_1(\sigma_1^*), y_1(\sigma_1^*)) \\ &+ £^{-1}(g_1(\sigma_1^*)), \\ y_1(\sigma_1^*) &= \Phi_1(\sigma_1^*) - £^{-1}(Rx_1(\sigma_1^*)) \\ &-£^{-1}(Ry_1(\sigma_1^*)) \\ &-£^{-1}(N_1(r, y_1(\sigma_1^*))) + \\ &£^{-1}(h_1(\sigma_1^*)).\end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned}x_2(\sigma_1^*) &= \Psi_2(\sigma_1^*) - £^{-1}(Rx_2(\sigma_1^*)) \\ &-£^{-1}(Ry_2(\sigma_1^*)) \\ &-£^{-1}(N_2(r, £^{-1}(x_2(\sigma_1^*)), y_2(\sigma_1^*))) \\ &+ £^{-1}(g_2(\sigma_1^*)), \\ y_2(\sigma_1^*) &= \Phi_2(\sigma_1^*) - £^{-1}(Rx_2(\sigma_1^*)) \\ &-£^{-1}(Ry_2(\sigma_1^*)) \\ &-£^{-1}(N_2(r, £^{-1}(x_2(\sigma_1^*)), y_2(\sigma_1^*))) + \\ &£^{-1}(h_2(\sigma_1^*)),\end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned}x_1(\sigma_2^*) &= \Psi_1(\sigma_2^*) - £^{-1}(Rx_1(\sigma_2^*)) \\ &-£^{-1}(Ry_1(\sigma_2^*)) \\ &-£^{-1}(N_1(r, x_1(\sigma_2^*), y_1(\sigma_2^*))) + \\ &£^{-1}(g_1(\sigma_2^*)), \\ y_1(\sigma_2^*) &= \Phi_1(\sigma_2^*) - £^{-1}(Rx_1(\sigma_2^*)) - \\ &£^{-1}(Ry_1(\sigma_2^*)) \\ &-£^{-1}(N_1(r, x_1(\sigma_2^*), y_1(\sigma_2^*))) + \\ &£^{-1}(h_1(\sigma_2^*)),\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}x_2(\sigma_2^*) &= \Psi_2(\sigma_2^*) - £^{-1}(Rx_2(\sigma_2^*)) - \\ &£^{-1}(Ry_2(\sigma_2^*)) \\ &-£^{-1}(N_2(r, x_2(\sigma_2^*), y_2(\sigma_2^*))) + \\ &£^{-1}(g_2(\sigma_2^*)), \\ y_2(\sigma_2^*) &= \Phi_2(\sigma_2^*) - £^{-1}(Rx_2(\sigma_2^*)) - \\ &£^{-1}(Ry_2(\sigma_2^*)) \\ &-£^{-1}(N_2(r, x_2(\sigma_2^*), y_2(\sigma_2^*))) + \\ &£^{-1}(h_2(\sigma_2^*)),\end{aligned}\right\} \quad (11)$$

where,

$$\left.\begin{aligned}\Psi_i(\sigma_1^*) &= L\Psi_i(\sigma_1^*) = 0, i = 1, 2 \\ \Phi_i(\sigma_1^*) &= L\Phi_i(\sigma_1^*) = 0, i = 1, 2\end{aligned}\right\}$$

$$\left.\begin{aligned}\Psi_i(\sigma_2^*) &= L\Psi_i(\sigma_2^*) = 0, i = 1, 2 \\ \Phi_i(\sigma_2^*) &= L\Phi_i(\sigma_2^*) = 0, i = 1, 2\end{aligned}\right\}$$

the above functions are found by using the initial conditions. Now by using the GMADM the solutions of the (8)−(11), can be expressed in the form of an infinite series for the unknown functions as follows:

$$\left.\begin{aligned}x_1(\sigma_1^*) &= \sum_{n=0}^{\infty} x_{1_n}(\sigma_1^*), \\ y_1(\sigma_1^*) &= \sum_{n=0}^{\infty} y_{1_n}(\sigma_1^*),\end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned}x_2(\sigma_1^*) &= \sum_{n=0}^{\infty} x_{2_n}(\sigma_1^*), \\ y_2(\sigma_1^*) &= \sum_{n=0}^{\infty} y_{2_n}(\sigma_1^*),\end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}x_1(\sigma_2^*) &= \sum_{n=0}^{\infty} x_{1_n}(\sigma_2^*), \\ y_1(\sigma_2^*) &= \sum_{n=0}^{\infty} y_{1_n}(\sigma_2^*),\end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned}x_2(\sigma_2^*) &= \sum_{n=0}^{\infty} x_{2_n}(\sigma_2^*), \\ y_2(\sigma_2^*) &= \sum_{n=0}^{\infty} y_{2_n}(\sigma_2^*),\end{aligned}\right\} \quad (15)$$

Using (12)−(15), in (8)−(11), we have:

$$\left.\begin{aligned}&\sum_{n=0}^{\infty} x_{1_n}(\sigma_1^*) = \Psi_1(\sigma_1^*) - \\ &£^{-1}\left(R\sum_{n=0}^{\infty} x_{1_n}(\sigma_1^*)\right) \\ &-£^{-1}\left(R\sum_{n=0}^{\infty} y_{1_n}(\sigma_1^*)\right) - \\ &£^{-1}\left(N_1\left(r, \sum_{n=0}^{\infty} x_{1_n}(\sigma_1^*),\right.\right. \\ &\left.\left.\sum_{n=0}^{\infty} y_{1_n}(\sigma_1^*)\right)\right) + £^{-1}(g_1(\sigma_1^*)), \\ &\sum_{n=0}^{\infty} y_{1_n}(\sigma_1^*) = \Phi_1(\sigma_1^*) - \\ &£^{-1}\left(R\sum_{n=0}^{\infty} x_{1_n}(\sigma_1^*)\right) \\ &-£^{-1}\left(R\sum_{n=0}^{\infty} y_{1_n}(\sigma_1^*)\right) - \\ &£^{-1}\left(N_1\left(r, \sum_{n=0}^{\infty} x_{1_n}(\sigma_1^*),\right.\right. \\ &\left.\left.\sum_{n=0}^{\infty} y_{1_n}(\sigma_1^*)\right)\right) + £^{-1}(h_1(\sigma_1^*)),\end{aligned}\right\} \quad (16)$$

$$\left.\begin{array}{c}\sum_{n=0}^{\infty} x_{2_n}(\sigma_1^*) = \Psi_2(\sigma_1^*)- \\ £^{-1}\left(R\sum_{n=0}^{\infty} x_{2_n}(\sigma_1^*)\right) \\ -£^{-1}\left(R\sum_{n=0}^{\infty} y_{2_n}(\sigma_1^*)\right)- \\ £^{-1}\left(N_2\left(r,\ \sum_{n=0}^{\infty} x_{2_n}(\sigma_1^*),\right.\right. \\ \left.\left.\sum_{n=0}^{\infty} y_{2_n}(\sigma_1^*)\right)\right) + £^{-1}(g_2(\sigma_1^*)), \\ \sum_{n=0}^{\infty} y_{2_n}(\sigma_1^*) = \Phi_2(\sigma_1^*)- \\ £^{-1}\left(R\sum_{n=0}^{\infty} x_{2_n}(\sigma_1^*)\right) \\ -£^{-1}\left(R\sum_{n=0}^{\infty} y_{2_n}(\sigma_1^*)\right)- \\ £^{-1}\left(N_1\left(r,\ \sum_{n=0}^{\infty} x_{2_n}\right)\sigma_1^*\right), \\ \sum_{n=0}^{\infty} y_{2_n}(\sigma_1^*))) + £^{-1}(h_2(\sigma_1^*)), \end{array}\right\} \quad (17)$$

$$\left.\begin{array}{c}\sum_{n=0}^{\infty} x_{1_n}(\sigma_2^*) = \Psi_1(\sigma_2^*)- \\ £^{-1}\left(R\sum_{n=0}^{\infty} x_{1_n}(\sigma_2^*)\right) \\ -£^{-1}\left(R\sum_{n=0}^{\infty} y_{1_n}(\sigma_2^*)\right)- \\ £^{-1}\left(N_1\left(r,\ \sum_{n=0}^{\infty} x_{1_n}\right)\sigma_2^*\right), \\ \sum_{n=0}^{\infty} y_{1_n}(\sigma_2^*))) + £^{-1}(g_1(\sigma_2^*)), \\ \sum_{n=0}^{\infty} y_{1_n}(\sigma_2^*) = \Phi_1(\sigma_2^*)- \\ £^{-1}\left(R\sum_{n=0}^{\infty} x_{1_n}(\sigma_2^*)\right) \\ -£^{-1}\left(R\sum_{n=0}^{\infty} y_{1_n}(\sigma_2^*)\right)- \\ £^{-1}\left(N_1\left(r,\ \sum_{n=0}^{\infty} x_{1_n}(\sigma_2^*),\right.\right. \\ \left.\left.\sum_{n=0}^{\infty} y_{1_n}(\sigma_2^*)\right)\right) + £^{-1}(h_1(\sigma_2^*)), \end{array}\right\} \quad (18)$$

$$\left.\begin{array}{c}\sum_{n=0}^{\infty} x_{2_n}(\sigma_2^*) = \Psi_2(\sigma_2^*)- \\ £^{-1}\left(R\sum_{n=0}^{\infty} x_{2_n}\left(\sigma_2^*\right)\right) \\ -£^{-1}\left(R\sum_{n=0}^{\infty} y_{2_n}\left(\sigma_2^*\right)\right)- \\ £^{-1}\left(N_2\left(r,\ \sum_{n=0}^{\infty} x_{2_n}(\sigma_2^*),\right.\right. \\ \left.\left.\sum_{n=0}^{\infty} y_{2_n}(\sigma_2^*)\right)\right) + £^{-1}(g_2(\sigma_2^*)), \\ \sum_{n=0}^{\infty} y_{2_n}(\sigma_2^*) = \Phi_2(\sigma_2^*)- \\ £^{-1}\left(R\sum_{n=0}^{\infty} x_{2_n}\left(\sigma_2^*\right)\right) \\ -£^{-1}\left(R\sum_{n=0}^{\infty} y_{2_n}\left(\sigma_2^*\right)\right)- \\ £^{-1}(N_1(r,\ \sum_{n=0}^{\infty} x_{2_n}(\sigma_2^*), \\ \sum_{n=0}^{\infty} y_{2_n}(\sigma_2^*))) + £^{-1}(h_2(\sigma_2^*)). \end{array}\right\} \quad (19)$$

According to the GMADM the recursive relation for the above expression are as follows:

$$\left.\begin{array}{c} x_{1_0}(\sigma_1^*) = \Psi_1(\sigma_1^*), \\ y_{1_0}(\sigma_1^*) = \Phi_1(\sigma_1^*), \\ x_{1_1}(\sigma_1^*) = £^{-1}(g_1(\sigma_1^*)) \\ -£^{-1}(Rx_{1_0}(\sigma_1^*)) \\ -£^{-1}(Ry_{1_0}(\sigma_1^*))- \\ £^{-1}(N_1(r, x_{1_0}(\sigma_1^*), y_{1_0}(\sigma_1^*))), \\ y_{1_1}(\sigma_1^*) = £^{-1}(h_1(\sigma_1^*))- \\ £^{-1}(Rx_{1_0}(\sigma_1^*)) \\ -£^{-1}(Ry_{1_0}(\sigma_1^*))- \\ £^{-1}(N_1(r, x_{1_0}(\sigma_1^*), y_{1_0}(\sigma_1^*))), \\ x_{1_{k+1}}(\sigma_1^*) = -£^{-1}(Rx_{1_k}(\sigma_1^*))- \\ £^{-1}(Ry_{1_k}(\sigma_1^*)) \\ -£^{-1}(N_1(r, x_{1_k}(\sigma_1^*), y_{1_k}(\sigma_1^*))), k \geqslant 1, \\ y_{1_{k+1}}(\sigma_1^*) = -£^{-1}(Rx_{1_k}(\sigma_1^*))- \\ £^{-1}(Ry_{1_k}(\sigma_1^*)) \\ -£^{-1}(N_1(r, x_{1k}(\sigma_1^*), y_{1_k}(\sigma_1^*))), k \geqslant 1, \end{array}\right\} \quad (20)$$

$$\left.\begin{array}{c} x_{2_0}(\sigma_1^*) = \Psi_2(\sigma_1^*), \\ y_{2_0}(\sigma_1^*) = \Phi_2(\sigma_1^*), \\ x_{2_1}(\sigma_1^*) = £^{-1}(g_2(\sigma_1^*)) - £^{-1}(Rx_{2_0}(\sigma_1^*)) \\ -£^{-1}(Ry_{2_0}(\sigma_1^*)) - £^{-1}(N_2(r, x_{2_0}(\sigma_1^*), y_{2_0}(\sigma_1^*))), \\ y_{2_1}(\sigma_1^*) = £^{-1}(h_2(\sigma_1^*)) - £^{-1}(Rx_{2_0}(\sigma_1^*)) \\ -£^{-1}(Ry_{2_0}(\sigma_1^*)) - £^{-1}(N_2(r, x_{2_0}(\sigma_1^*), y_{2_0}(\sigma_1^*))), \\ x_{2_{k+1}}(\sigma_1^*) = -£^{-1}(Rx_{2k}(\sigma_1^*)) - £^{-1}(Ry_{2_k}(\sigma_1^*)) \\ -£^{-1}(N_2(r, x_{2_k}(\sigma_1^*), y_{2k}(\sigma_1^*))), k \geqslant 1, \\ y_{2_{k+1}}(\sigma_1^*) = -£^{-1}(Rx_{2_k}(\sigma_1^*)) - £^{-1}(Ry_{2_k}(\sigma_1^*)) \\ -£^{-1}(N_2(r, x_{2_k}(\sigma_1^*), y_{2k}(\sigma_1^*))), k \geqslant 1, \end{array}\right\} \quad (21)$$

$$\left.\begin{array}{c} x_{1_0}(\sigma_2^*) = \Psi_1(\sigma_2^*), \\ y_{1_0}(\sigma_2^*) = \Phi_1(\sigma_2^*), \\ x_{1_1}(\sigma_2^*) = £^{-1}(g_1(\sigma_2^*)) - \\ £^{-1}(Rx_{1_0}(\sigma_2^*)) \\ -£^{-1}(Ry_{1_0}(\sigma_2^*)) - \\ £^{-1}(N_1(r, x_{1_0}(\sigma_2^*), y_{1_0}(\sigma_2^*))), \\ y_{1_1}(\sigma_2^*) = £^{-1}(h_1(\sigma_2^*)) - \\ £^{-1}(Rx_{1_0}(\sigma_2^*)) \\ -£^{-1}(Ry_{1_0}(\sigma_2^*)) - \\ £^{-1}(N_1(r, x_{1_0}(\sigma_2^*), y_{1_0}(\sigma_2^*))), \\ x_{1_{k+1}}(\sigma_2^*) = -£^{-1}(Rx_{1_k}(\sigma_2^*)) - \\ £^{-1}(Ry_{1_k}(\sigma_2^*)) \\ -£^{-1}(N_1(r, x_{1_k}(\sigma_2^*), y_{1_k}(\sigma_2^*))), k \geqslant 1, \\ y_{1_{k+1}}(\sigma_2^*) = -£^{-1}(Rx_{1_k}(\sigma_2^*)) - \\ £^{-1}(Ry_{1_k}(\sigma_2^*)) \\ -£^{-1}(N_1(r, x_{1_k}(\sigma_2^*), y_{1_k}(\sigma_2^*))), k \geqslant 1, \end{array}\right\} \quad (22)$$

$$\left.\begin{array}{c} x_{2_0}(\sigma_2^*) = \Psi_2(\sigma_2^*), \\ y_{2_0}(\sigma_2^*) = \Phi_2(\sigma_2^*), \\ x_{2_1}(\sigma_2^*) = £^{-1}(g_2(\sigma_2^*)) - \\ £^{-1}(Rx_{2_0}(\sigma_2^*)) \\ -£^{-1}(Ry_{2_0}(\sigma_2^*)) - \\ £^{-1}(N_2(r, x_{2_0}(\sigma_2^*), y_{2_0}(\sigma_2^*))), \\ y_{2_1}(\sigma_2^*) = £^{-1}(h_2(\sigma_2^*)) - \\ £^{-1}(Rx_{2_0}(\sigma_2^*)) \\ -£^{-1}(Ry_{2_0}(\sigma_2^*)) - \\ £^{-1}(N_2(r, x_{2_0}(\sigma_2^*), y_{2_0}(\sigma_2^*))), \\ x_{2_{k+1}}(\sigma_2^*) = -£^{-1}(Rx_{2_k}(\sigma_2^*)) - \\ £^{-1}(Ry_{2_k}(\sigma_2^*)) \\ -£^{-1}(N_2(r, x_{2_k}(\sigma_2^*), y_{2_k}(\sigma_2^*))), k \geqslant 1, \\ y_{2_{k+1}}(\sigma_2^*) = -£^{-1}(Rx_{2_k}(\sigma_2^*)) \\ -£^{-1}(Ry_{2_k}(\sigma_2^*)) \\ -£^{-1}(N_2(r, x_{2_k}(\sigma_2^*), y_{2_k}(\sigma_2^*))), k \geqslant 1. \end{array}\right\} \quad (23)$$

The nth-term approximation to the solution is defined as follows:

$$\left.\begin{array}{l} \varphi_{1n}(\sigma_1^*) = \sum_{i=0}^{n-1} x_{1_i}(\sigma_1^*), \\ \phi_{1n}(\sigma_1^*) = \sum_{i=0}^{n-1} y_{1_i}(\sigma_1^*), \end{array}\right\} \quad (24)$$

$$\left.\begin{array}{l} \varphi_{2n}(\sigma_1^*) = \sum_{i=0}^{n-1} x_{2_i}(\sigma_1^*), \\ \phi_{2n}(\sigma_1^*) = \sum_{i=0}^{n-1} y_{2_i}(\sigma_1^*), \end{array}\right\} \quad (25)$$

$$\left.\begin{array}{l} \varphi_{1n}(\sigma_2^*) = \sum_{i=0}^{n-1} x_{1_i}(\sigma_2^*), \\ \phi_{1n}(\sigma_2^*) = \sum_{i=0}^{n-1} y_{1_i}(\sigma_2^*), \end{array}\right\} \quad (26)$$

$$\left.\begin{array}{l} \varphi_{2n}(\sigma_2^*) = \sum_{i=0}^{n-1} x_{2_i}(\sigma_2^*), \\ \phi_{1n}(\sigma_2^*) = \sum_{i=0}^{n-1} y_{1_i}(\sigma_2^*). \end{array}\right\} \quad (27)$$

Hence,

$$\left\{\begin{array}{l} \{\lim_{n\to\infty}(\phi_{1n}(\sigma_1^*)), \lim_{n\to\infty} \phi_{1n}(\sigma_1^*)\} \\ \quad = \{x_1(\sigma_1^*), y_1(\sigma_1^*)\}, \\ \{\lim_{n\to\infty}(\phi_{2n}(\sigma_1^*)), \lim_{n\to\infty} \phi_{2n}(\sigma_1^*)\} \\ \quad = \{x_2(\sigma_1^*), y_2(\sigma_1^*)\}, \\ \{\lim_{n\to\infty}(\phi_{1n}(\sigma_2^*)), \lim_{n\to\infty} \phi_{1n}(\sigma_2^*)\} \\ \quad = \{x_1(\sigma_2^*), y_1(\sigma_2^*)\}, \\ \{\lim_{n\to\infty}(\phi_{2n}(\sigma_2^*)), \lim_{n\to\infty} \phi_{2n}(\sigma_2^*)\} \\ \quad = \{x_2(\sigma_2^*), y_2(\sigma_2^*)\}. \end{array}\right. \quad (28)$$

Numerical outcomes

In this section, we show some examples to illustrate the effectiveness and efficiency of the GMADM method, Taylor's series method (GTM) and generalize decomposition method (GDM) for solving triangular fuzzy system of intuitionistic differential equations. The computer application is terminated using CAS- MatLab 2011Rb and used the following terminating criteria

$$\left\{\begin{array}{l} \frac{|x_1^*(\sigma_1^*) - x_1(\sigma_1^*)|}{|x_1^*(\sigma_1^*)|} \\ \frac{|y_1^*(\sigma_1^*) - y_1(\sigma_1^*)|}{|y_1^*(\sigma_1^*)|} \end{array}\right.$$

$$\left\{\begin{array}{l} \frac{|x_2^*(\sigma_2^*) - x_2(\sigma_2^*)|}{|x_2^*(\sigma_2^*)|} \\ \frac{|y_2^*(\sigma_2^*) - y_2(\sigma_2^*)|}{|y_2^*(\sigma_2^*)|} \end{array}\right.$$

where $x_1^*(\sigma_1^*), x_2^*(\sigma_2^*), y_1^*(\sigma_1^*), y_2^*(\sigma_2^*)$ are the corresponding exact solutions of $x_1(\sigma_1^*), x_2(\sigma_2^*), y_1(\sigma_1^*), y_2(\sigma_2^*)$ respectively. The complexity of the GMADM algorithm is approximately $O(Nlog(N))$, according to numerical results using two or more Fuzzy systems of intuitionistic differential equations.

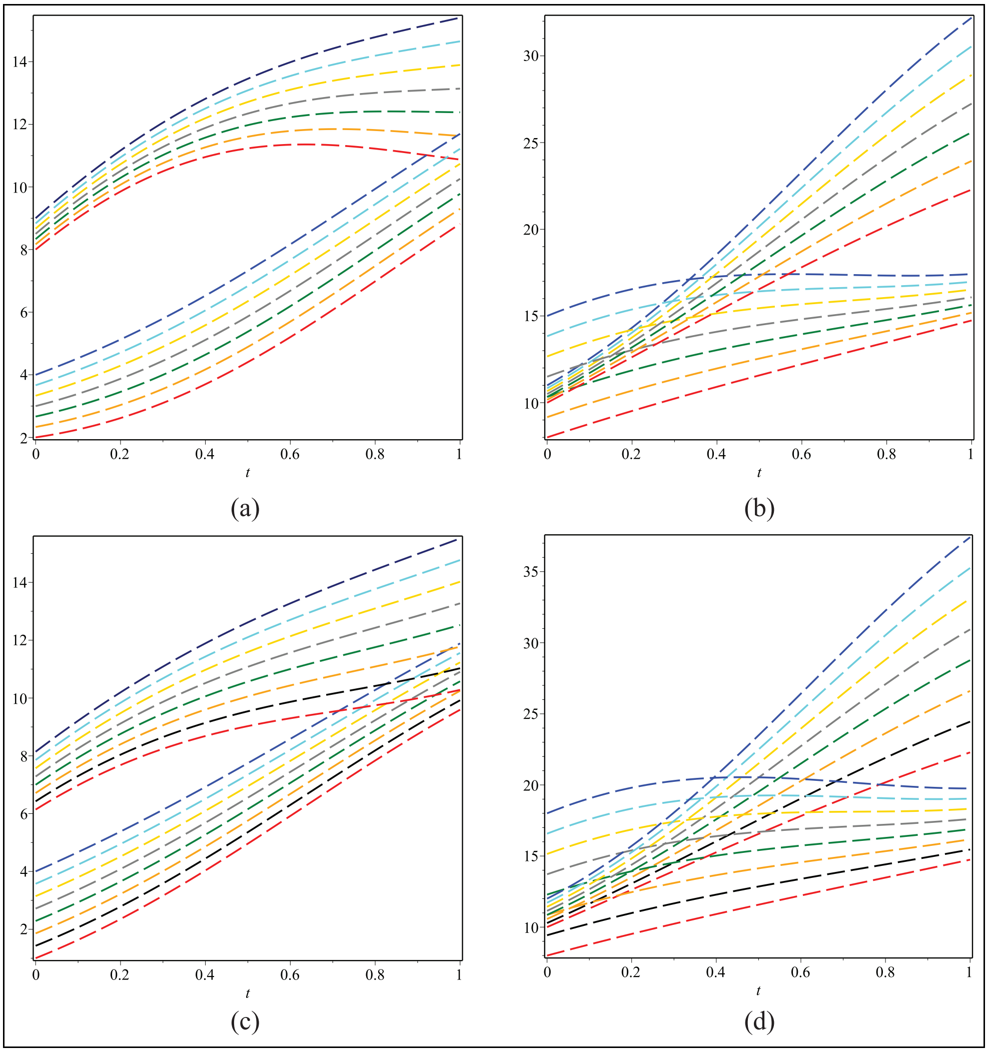

Figure 2. (a–d): clearly shows that numerical approximate solution obtained by GMADM are exactly matched with analytical solution of system of intuitionistic triangular fuzzy initial value problems used in Example 1. (a) Numerical and analytical solution of x_1, (b) numerical and analytical solution of x_2, (c) numerical and analytical solution of x_3, and (d) numerical and analytical solution of TIFLSEs used in Example 1.

Engineering applications

Here, we discuss some engineering applications which contains system of intuitionistic fuzzy differential equations.

Example 1: Mechanical Engineering Problem:

Consider a mechanical system[30,32,33] that consist of two masses, m1 and m2, each of which is free to slide along a horizontal surface with no fraction. Three springs are used to bind the masses to each other as well as to two stiff walls. The system's spring constants are k1 = 1 Nm, K2 = 2 Nm, and K3 = 1 Nm. The $x(t)$ and $y(t)$, respectively, conveniently specify the system current state at any given time. The following are the equations for the motion of two masses rise the following first order system of non-linear homogeneous intuitionistic fuzzy differential equations are as follows:

$$\begin{cases} m\frac{d^2x(t)}{dt^2} = -k_1x(t) - k_2(x(t) - y(t)), \\ m\frac{d^2y(t)}{dt^2} = -k_3y(t) - k_2(y(t) - x(t)). \end{cases} \tag{29}$$

Using the given data, we get;

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x(t) + 3y(t) - 7, \\ \frac{dy}{dt} &= -x(t) - 2y(t) + 5. \end{aligned}\right\} \tag{30}$$

with initial conditions

$$\left.\begin{aligned} x(0) &= <2, 4, 8; 1, 4, 9>, \\ y(0) &= <2, 5, 8; 1, 5, 9>. \end{aligned}\right\} \tag{31}$$

By taking $(\vartheta_1, \vartheta_2)$-cuts (30), we get;

$$\left.\begin{aligned}\frac{dx_1(\sigma_1^*)}{dt} &= 2x_1(\sigma_1^*) + 3y_1(\sigma_1^*) - 7,\\ x_1(\sigma_{01}^*) &= 2\vartheta_1 + 2,\\ \frac{dy_1(\sigma_1^*)}{dt} &= -x_1(\sigma_1^*) - 2y_1(\sigma_1^*) + 5,\\ y_1(\sigma_{01}^*) &= 3\vartheta_1 + 2,\end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned}\frac{dx_2(\sigma_1^*)}{dt} &= 2x_2(\sigma_1^*) + 3y_2(\sigma_1^*) - 7,\\ x_2(\sigma_{01}^*) &= -4\vartheta_1 + 8,\\ \frac{dy_2(\sigma_1^*)}{dt} &= -x_2(\sigma_1^*) - 2y_2(\sigma_1^*) + 5,\\ y_2(\sigma_{01}^*) &= -3\vartheta_1 + 8,\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned}\frac{dx_1(\sigma_2^*)}{dt} &= 2x_1(\sigma_2^*) + 3y_1(\sigma_2^*) - 7,\\ x_1(\sigma_{02}^*) &= -3\vartheta_2 + 4,\\ \frac{dy_1(\sigma_2^*)}{dt} &= -x_1(\sigma_2^*) - 2y_1(\sigma_2^*) + 5,\\ y_1(\sigma_{02}^*) &= -4\vartheta_2 + 5,\end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned}\frac{dx_2(\sigma_2^*)}{dt} &= 2x_2(\sigma_2^*) + 3y_2(\sigma_2^*) - 7,\\ x_2(\sigma_{02}^*) &= 5\vartheta_2 + 4,\\ \frac{dy_2(\sigma_2^*)}{dt} &= -x_2(\sigma_2^*) - 2y_2(\sigma_2^*) + 5,\\ y_2(\sigma_{02}^*) &= 4\vartheta_2 + 5.\end{aligned}\right\} \tag{35}$$

where

$$\sigma_{01}^* = (0, \vartheta); \sigma_{02}^* = (0, \beta);$$

and . Here $\pounds = \frac{d}{dt}$ and by taking $\pounds^{-1}(.) = \int_0^t (.)dt$ on both sides of (32)−(35), and using the initial conditions we obtain;

$$\begin{cases} x_1(\sigma_1^*) = \int_0^t (2x_1(\sigma_1^*) + 3y_1(\sigma_1^*))dt + 2 + 2\vartheta_1 - 7r,\\ y_1(\sigma_1^*) = \int_0^t (-x_1(\sigma_1^*) - 2y_1(\sigma_1^*))dt + 2 + 3\vartheta_1 + 5r,\end{cases} \tag{36}$$

$$\begin{cases} x_2(\sigma_1^*) = \int_0^t (2x_2(\sigma_1^*) + 3y_2(\sigma_1^*))dt + 8 - 4\vartheta_1 - 7r,\\ y_2(\sigma_1^*) = \int_0^t (-x_2(\sigma_1^*) - 2y_2(\sigma_1^*))dt + 8 - 3\vartheta_1 + 5r,\end{cases} \tag{37}$$

$$\begin{cases} x_1(\sigma_2^*) = \int_0^t (2x_1(u, \vartheta_2) + 3y_1(u, \vartheta_2))du + 4 - 3\vartheta_2 - 7t,\\ y_1(\sigma_2^*) = \int_0^t (-x_1(u, \vartheta_2) - 2y_1(u, \vartheta_2))du + 5 - 4\vartheta_2 + 5t,\end{cases} \tag{38}$$

$$\begin{cases} x_2(\sigma_2^*) = \int_0^t (2x_2(u, \vartheta_2) + 3y_2(u, \vartheta_2))du + 4 + 5\vartheta_2 - 7t,\\ y_2(\sigma_2^*) = \int_0^t (-x_2(u, \vartheta_2) - 2y_2(u, \vartheta_2))du + 4 + 5\vartheta_2 + 5t.\end{cases} \tag{39}$$

Now by using GMADM we get;

$$\begin{cases} x_{1_0}(\sigma_1^*) = 2\vartheta_1 + 2,\\ y_{1_0}(\sigma_1^*) = 3\vartheta_1 + 2,\\ x_{1_1}(\sigma_1^*) = 13\vartheta_1 + 3t,\\ y_{1_1}(\sigma_1^*) = -8\vartheta_1 - t,\\ x_{1_{k+1}}(\sigma_1^*) = \int_0^u \begin{pmatrix} 2x_{1k}(u, \vartheta_1) + \\ 3y_{1k}(u, \vartheta_1)\end{pmatrix} du, k \geqslant 1,\\ y_{1_{k+1}}(\sigma_1^*) = \int_0^t \begin{pmatrix} -x_{1k}(u, \vartheta_1) - \\ 2y_{1_k}(u, \vartheta_1)\end{pmatrix} du, k \geqslant 1.\end{cases} \tag{40}$$

$$\begin{cases} x_{2_0}(\sigma_1^*) = -4\vartheta_1 + 8,\\ y_{2_0}(\sigma_1^*) = -3\vartheta_1 + 8,\\ x_{2_1}(\sigma_1^*) = -17\vartheta_1 + 33t,\\ y_{2_1}(\sigma_1^*) = 10\vartheta_1 - 19t,\\ x_{2_{k+1}}(\sigma_1^*) = \int_0^u (2x_{2k}(u, \vartheta_1) + 3y_{2k}(u, \vartheta_1))du, k \geqslant 1,\\ y_{2_{k+1}}(\sigma_1^*) = \int_0^u (-x_{2k}(u, \vartheta_1) - 2y_{2_k}(u, \vartheta_1))du, k \geqslant 1,\end{cases} \tag{41}$$

$$\begin{cases} x_{1_0}(\sigma_2^*) = 4 - 3\vartheta_2,\\ y_{1_0}(\sigma_2^*) = 5 - 4\vartheta_2,\\ x_{1_1}(\sigma_2^*) = 16t - 18\vartheta_2,\\ y_{1_1}(\sigma_2^*) = -9t + 11\vartheta_2,\\ x_{1_{k+1}}(\sigma_2^*) = \int_0^u \begin{pmatrix} 2x_{1k}(u, \vartheta_1) + \\ 3y_{1k}(u, \vartheta_1)\end{pmatrix} du, k \geqslant 1,\\ y_{1_{k+1}}(\sigma_2^*) = \int_0^u \begin{pmatrix} -x_{1k}(u, \vartheta_1) - \\ 2y_{1_k}(u, \vartheta_1)\end{pmatrix} du, k \geqslant 1.\end{cases} \tag{42}$$

$$\begin{cases} x_{2_0}(\sigma_2^*) = 4 + 5\vartheta_2,\\ y_{2_0}(\sigma_2^*) = 4 + 5\vartheta_2,\\ x_{2_1}(\sigma_2^*) = 13t + 25\vartheta_2,\\ y_{2_1}(\sigma_2^*) = -7t - 15\vartheta_2,\\ x_{2_{k+1}}(\sigma_2^*) = \int_0^u \begin{pmatrix} 2x_{2k}(u, \vartheta_1) + \\ 3y_{2k}(u, \vartheta_1)\end{pmatrix} du, k \geqslant 1,\\ y_{2_{k+1}}(\sigma_2^*) = \int_0^r \begin{pmatrix} -x_{2k}(u, \vartheta_1) - \\ 2y_{2_k}(u, \vartheta_1)\end{pmatrix} du, k \geqslant 1.\end{cases} \tag{43}$$

By solving the (39)−(43), we get the approximate solution after four iterations as follows:

$$\begin{aligned}&(x_1(\sigma_1^*), y_1(\sigma_1^*))\\ &= \Big(2\vartheta_1 + 2 + 13\vartheta_1 t + 3t + t^2\vartheta_1 + \frac{3}{2}t^2 +\\ &\frac{13}{6}t^3\vartheta_1 + \frac{1}{2}t^3 + \frac{1}{12}t^4\vartheta_1 + \frac{1}{8}t^4,\\ &3\vartheta_1 + 2 - 8\vartheta_1 t -\\ &t + \frac{3}{2}t^2\vartheta_1 - \frac{1}{2}t^2 - \frac{4}{3}t^3\vartheta_1 - \frac{1}{6}t^3 +\\ &\frac{1}{8}t^4\vartheta_1 - \frac{1}{24}t^4\Big),\end{aligned} \tag{44}$$

$$\begin{aligned}&(x_2(\sigma_1^*), y_2(\sigma_1^*))\\&=\Big(-4\vartheta_1+8-17\vartheta_1 t+33t-2t^2\vartheta_1+\\&\frac{9}{2}t^2-\frac{17}{6}t^3\vartheta_1+\frac{11}{2}t^3\\&-\frac{1}{6}t^4\vartheta_1+\frac{3}{8}t^4,\ -3\vartheta_1+8+10\vartheta_1 t-\\&19t-\frac{3}{2}t^2\vartheta_1+\frac{5}{2}t^2+\frac{5}{3}t^3\vartheta_1-\frac{19}{6}t^3\\&-\frac{1}{8}t^4\vartheta_1+\frac{5}{24}t^4\Big),\end{aligned} \tag{45}$$

$$\begin{aligned}&(x_1(\sigma_2^*), y_1(\sigma_2^*))\\&=\Big(4-3\vartheta_2-18\vartheta_2 t+16t-\frac{3}{2}t^2\vartheta_2+\\&\frac{5}{2}t^2-3t^3\vartheta_2+\frac{8}{3}t^3-\frac{1}{8}t^4\vartheta_2+\frac{5}{24}t^4,\\&5-4\beta+11\vartheta_2 t-9t-2t^2\vartheta_2-\\&t^2\vartheta_2+t^2+\frac{11}{6}t^3\vartheta_2-\frac{3}{2}t^3-\frac{1}{6}t^4\vartheta_2+\frac{1}{12}t^4\Big),\end{aligned} \tag{46}$$

$$\begin{aligned}&(x_2(\sigma_2^*), y_2(\sigma_2^*))\\&=\Big(5\vartheta_2+4+25\vartheta_2 t+13t+\frac{5}{2}t^2\vartheta_2+\\&\frac{5}{2}t^2+\frac{25}{6}t^3\vartheta_2+\frac{13}{6}t^3\\&+\frac{5}{24}t^4\vartheta_2+\frac{5}{24}t^4, 5\vartheta_2+4-15\vartheta_2 t-\\&7t+\frac{5}{2}t^2\vartheta_2+\frac{1}{2}t^2-\frac{5}{2}t^3\vartheta_2-\\&\frac{7}{6}t^3+\frac{5}{24}t^4\vartheta_2+\frac{1}{24}t^4\Big).\end{aligned} \tag{47}$$

In Table 1, $x_1(\sigma_1^*)$,$x_2(\sigma_1^*)$,$y_1(\sigma_1^*)$ and $y_2(\sigma_1^*)$ represents analytical solution of the membership functions of the Example 1 for $\vartheta_1 \in [0, 1.0]$.

In Table 2, $x_1(\sigma_2^*)$, $x_2(\sigma_2^*)$,$y_1(\sigma_2^*)$, and $y_2(\sigma_2^*)$ represents analytical solution of non-membership function of Example 1 for $\vartheta_2 \in [0, 1.0]$.

The exact solution of numerical example given by classical method is as follows:

$$\begin{cases} x_1(\sigma_1^*) = e^t\left(3+\frac{15}{2}\vartheta_1\right)-\frac{11}{2}e^{-t}\vartheta_1-1,\\ y_1(\sigma_1^*) = -\frac{1}{3}e^t\left(3+\frac{15}{2}\vartheta_1\right)+\frac{11}{2}e^{-t}\vartheta_1+3 \end{cases}, \tag{48}$$

$$\begin{cases} x_2(\sigma_1^*) = e^{-t}\left(-12+\frac{13}{2}\vartheta_1\right)+e^t\left(21-\frac{21}{2}\vartheta_1\right)-1,\\ y_2(\sigma_1^*) = -e^{-t}\left(-12+\frac{13}{2}\vartheta_1\right)-\frac{1}{3}\\ e^t\left(21-\frac{21}{2}\vartheta_1\right)+3 \end{cases}, \tag{49}$$

Table 1. The analytical solution for the membership functions of Example 1 for $\theta = 1$ that belong to [0, 1.0] after two iterations.

ϑ_1	x_1 (t, ϑ_1)	y_1 (t,ϑ_1)	x_2 (t,ϑ_1)	y_2 (t, ϑ_1)
0	7.125	0.292	51.38	−11.46
0.1	8.950	−0.179	48.78	−10.75
0.2	10.78	−0.650	46.18	−10.05
0.3	12.60	−1.121	43.58	−9.346
0.4	14.43	−1.592	40.98	−8.642
0.5	16.25	−2.063	38.38	−7.938
0.6	18.08	−2.533	35.78	−7.233
0.7	19.90	−3.004	33.18	−6.529
0.8	21.73	−3.475	30.58	−5.825
0.9	23.55	−3.946	27.98	−5.121
1.0	25.38	−4.417	25.38	−4.417

Table 2. The analytical solution for the non-membership functions of Example 1 for $\theta = 1$ that belong to [0, 1.0] after two iterations.

ϑ_2	x_1 (t,ϑ_2)	y_1 (t,ϑ_2)	x_2 (t,ϑ_2)	y_2 (t,ϑ_2)
0	25.38	4.42	21.88	−3.625
0.1	22.81	−3.750	25.56	−4.604
0.2	20.25	−3.083	29.25	−5.583
0.3	17.69	−2.417	32.94	−6.563
0.4	15.13	−1.750	36.63	−7.542
0.5	12.56	−1.083	40.31	−8.531
0.6	10.00	−0.417	44.00	−9.50
0.7	7.438	0.250	47.69	−10.48
0.8	4.875	0.917	51.38	−11.46
0.9	2.313	1.583	55.06	−12.44
1.0	−0.250	2.250	58.75	−13.42

$$\begin{cases} x_1(\sigma_2^*) = e^t\left(\frac{21}{2}-\frac{21}{2}\vartheta_2\right)+e^{-t}\left(-\frac{11}{2}+\frac{15}{2}\vartheta_2\right)-1,\\ y_1(\sigma_2^*) = -\frac{1}{3}e^t\left(\frac{21}{2}-\frac{21}{2}\vartheta_2\right)\\ -e^{-t}\left(-\frac{11}{2}+\frac{15}{2}\vartheta_2\right)+3 \end{cases}, \tag{50}$$

$$\begin{cases} x_2(\sigma_2^*) = e^{-t}\left(-\frac{11}{2}-\frac{17}{2}\vartheta_2\right)+e^t\left(\frac{21}{2}+\frac{27}{2}\vartheta_2\right)-1,\\ y_2(\sigma_2^*) = -e^{-t}\left(-\frac{11}{2}-\frac{17}{2}\vartheta_2\right)\\ -\frac{1}{3}e^t\left(\frac{21}{2}+\frac{27}{2}\vartheta_2\right)+3 \end{cases}. \tag{51}$$

In Table 3, $x_1(\sigma_1^*)$,$x_2(\sigma_1^*)$,$y_1(\sigma_1^*)$, and $y_2(\sigma_1^*)$ represents exact solution of the membership functions of Example 1 for $\vartheta_1 \in [0, 1.0]$.

In Table 4,$x_1(\sigma_2^*)$, $x_2(\sigma_2^*)$,$y_1(\sigma_2^*)$, and $y_2(\sigma_2^*)$ represents exact solution of non-membership function of Example 1 for $\vartheta_2 \in [0, 1.0]$.

The exact solution of Example 1 by classical method is given as follows:

Table 3. The exact solution to the membership functions for the $\theta = 1$ that belong to [0, 1.0] in Example 1.

ϑ_1	$x_1\ (t,\vartheta_1)$	$y_1\ (t,\vartheta_1)$	$x_2\ (t,\vartheta_1)$	$y_2\ (t,\vartheta_1)$
0	7.154	0.282	51.66	−11.62
0.1	8.988	−0.196	49.04	−10.89
0.2	10.83	−0.672	46.43	−10.18
0.3	12.66	−1.149	43.82	−9.47
0.4	14.50	−1.627	41.20	−8.76
0.5	16.34	−2.104	38.59	−8.05
0.6	18.17	−2.581	35.97	−7.34
0.7	20.00	−3.058	33.36	−6.629
0.8	21.84	−3.535	30.75	−5.918
0.9	23.68	−4.013	28.13	−5.197
1.0	25.52	−4.490	25.52	−4.490

Table 4. The exact solution to the non-membership functions for the $\theta = 1$ that belong to [0, 1] in Example 1.

ϑ_2	$x_1\ (t,\vartheta_2)$	$y_1\ (t,\vartheta_2)$	$x_2\ (t,\vartheta_2)$	$y_2\ (t,\vartheta_2)$
0	25.52	−4.490	25.52	−4.490
0.1	22.94	−3.814	28.87	−5.404
0.2	20.36	−3.138	32.23	−6.311
0.3	17.78	−2.463	35.59	−7.22
0.4	15.20	−1.788	38.95	−8.14
0.5	12.63	−1.112	42.30	−9.04
0.6	10.05	−0.437	45.65	−9.95
0.7	7.470	0.238	49.01	−10.86
0.8	4.892	0.913	52.36	−11.78
0.9	2.314	1.589	55.72	−12.68
1.0	−0.264	2.264	59.08	−13.59

$$(x_1(\sigma_1^*), y_1(\sigma_1^*)) \tag{52}$$

$$= \Big\{ -\frac{3}{5}sin(\sqrt{5}t)\sqrt{5} + \left(-9+\frac{1}{2}\vartheta_1\right)cos\left(\sqrt{5}t\right) + (11+\vartheta_1)sin(t) + \left(11+\frac{3}{2}\vartheta_1\right)cos(t), \tag{53}$$
$$\frac{3}{5}sin\left(\sqrt{5}t\right)\sqrt{5} - \left(-9+\frac{1}{2}\vartheta_1\right)cos\left(\sqrt{5}t\right) + (11+\vartheta_1)sin(t) + \left(11+\frac{3}{2}\vartheta_1\right)cos(t)\Big\},$$

$$(x_2(\sigma_1^*), y_2(\sigma_1^*)) = \Big\{ -\frac{3}{5}sin(\sqrt{5}t)\sqrt{5} - \frac{17}{2}cos(\sqrt{5}t) + (13-\vartheta_1)sin(t) + \left(\frac{27}{2}-\vartheta_1\right)cos(t), \tag{54}$$
$$\frac{3}{5}sin\left(\sqrt{5}t\right)\sqrt{5} + \frac{17}{2}cos\left(\sqrt{5}t\right) + (13-\vartheta_1)sin(t) + \left(\frac{27}{2}-\vartheta_1\right)cos(t)\Big\},$$

Table 5. A numerical comparison of GMADM, GTM, and GDM for solving the intuitionistic fuzzy system of differential equations used in Example 1.

Parameters	GMADM	GTM	GDM
CPU	0.0123	1.5143	1.0123
Ns	04	04	05
Err	$0.2e^{-13}$	$2.3e^{-2}$	$5.6e^{-4}$

$$(x_1(\sigma_2^*), y_1(\sigma_2^*)) = \Big\{(12-2\beta)sin(t) + \left(\frac{25}{2}-\frac{5}{2}\vartheta_2\right)cos(t) - \frac{3}{5}sin\left(\sqrt{5}t\right)\sqrt{5} + \left(-\frac{17}{2}-\frac{1}{2}\vartheta_2\right)cos\left(\sqrt{5}t\right), \tag{55}$$
$$(12-2\beta)sin(t) + \left(\frac{25}{2}-\frac{5}{2}\vartheta_2\right)cos(t) + \frac{3}{5}sin\left(\sqrt{5}t\right)\sqrt{5} - \left(-\frac{17}{2}-\frac{1}{2}\vartheta_2\right)cos\left(\sqrt{5}t\right)\Big\},$$

$$(x_2(\sigma_2^*), y_2(\sigma_2^*)) = \Big\{ -\frac{3}{5}sin(\sqrt{5}t)\sqrt{5} - \frac{17}{2}cos(\sqrt{5}t) + (12+2\beta)sin(t) + \left(\frac{25}{2}+2\beta\right)cos(t), \tag{56}$$
$$\frac{3}{5}sin\left(\sqrt{5}t\right)\sqrt{5} + \frac{17}{2}cos\left(\sqrt{5}t\right) + (12+2\beta)sin(t) + \left(\frac{25}{2}+2\beta\right)cos(t)\Big\}.$$

Table 5. shows numerical comparison of computational time seconds (CPU), number of iterations (Ns), and residual error (Err) of GMADM, GTM, and GDM respectively for solving system of intuitionistic fuzzy initial value problems used in engineering application.

Example 2: Consider the second order homogeneous system of intuitionistic fuzzy differential equations:

$$\begin{cases} \dfrac{d^2x(t)}{dt^2} = 2y - 3x, \\ \dfrac{d^2y(t)}{dt^2} = 2x - 3y. \end{cases} \tag{57}$$

with initial conditions,

$$\begin{cases} x(0) = <2,4,5;1,4,6>, \\ x'(0) = <8,9,10;7,9,11>, \\ y(0) = <20,21,22;19,21,23>, \\ y'(0) = <14,15,16;13,15,17>. \end{cases} \tag{58}$$

By taking $(\vartheta_1, \vartheta_2)$-cut of (58), we get the following equations:

$$\begin{cases} \frac{d^2x_1(\sigma_1^*)}{dt^2} = 2y_1(\sigma_1^*) - 3x_1(\sigma_1^*), \\ x_1(0,\vartheta) = 2\alpha + 2, x_1'(0,\vartheta) = \vartheta + 8, \\ \frac{d^2y_1(\sigma_1^*)}{dt^2} = 2x_1(\sigma_1^*) - 3y_1(\sigma_1^*), \\ y_1(0,\vartheta) = \vartheta + 20, y_1'(0,\vartheta) = \vartheta + 14 \end{cases} \tag{59}$$

$$\begin{cases} \frac{d^2x_2(\sigma_1^*)}{dt^2} = 2y_2(\sigma_1^*) - 3x_2(\sigma_1^*), \\ x_2(0,\vartheta) = 5 - \vartheta_1, x_2'(0,\vartheta_1) = 10 - \vartheta_1, \\ \frac{d^2y_2(\sigma_1^*)}{dt^2} = 2x_2(t,\vartheta_1) - 3y_2(t,\vartheta_1), \\ y_2(0,\vartheta_1) = -\vartheta_1 + 22, y_2'(0,\vartheta_1) = -\vartheta_1 + 16, \end{cases} \tag{60}$$

$$\begin{cases} \frac{d^2x_1(\sigma_2^*)}{dt^2} = 2\overset{\star}{y_1}(\sigma_2^*) - 3x_1(\sigma_2^*), \\ x_1(0,\vartheta_2) = -3\vartheta_2 + 4, x_1'(0,\vartheta_2) = 9 - 2v, \\ \frac{d^2y_1(\sigma_2^*)}{dt^2} = 2x_1(\sigma_2^*) - 3y_1(\sigma_2^*), \\ y_1(0,\vartheta_2) = 21 - 2\vartheta_2, y_1'(0,\vartheta_2) = 15 - 2\vartheta_2 \end{cases} \tag{61}$$

$$\begin{cases} \frac{d^2x_2(\sigma_2^*)}{dt^2} = 2y_2(\sigma_2^*) - 3x_2(\sigma_2^*), \\ x_2(0,\vartheta_2) = 4 + 2\vartheta_2, x_1'(0,\vartheta 2) = 9 + 2\vartheta_2, \\ \frac{d^2y_2(\sigma_2^*)}{dt^2} = 2x_2(\sigma_2^*) - 3y_2(\sigma_2^*), \\ y_2(0,\vartheta_2) = = 21 + 2\vartheta_2, y_1'(0,\vartheta_2) = 15 + 2\vartheta_2. \end{cases} \tag{62}$$

Here $\pounds = \frac{d^2}{dt^2}$ and by taking $\pounds^{-1}(.) = \int_0^t\int_0^t (.)dtdt$ on both sides of (60)−(62), and using the initial conditions, we obtain;

$$\begin{cases} x_1(\sigma_1^*) = \int_0^t \begin{pmatrix} 3x_1(u,\vartheta_1)(-t+u) + \\ 2y_1(u,\vartheta_1)(t-u) \end{pmatrix} du \\ \quad + \alpha t + 2\alpha + 20t + 2, \\ y_1(\sigma_1^*) = \int_0^t \begin{pmatrix} 2x_1(u,\vartheta_1)(t-u) + \\ 3y_1(u,\vartheta_1)(-t+u) \end{pmatrix} du \\ \quad \vartheta_1 t + \vartheta_1 + 14t + 8, \end{cases} \tag{63}$$

$$\begin{cases} x_2(\sigma_1^*) = \int_0^t \begin{pmatrix} 3x_2(u,\vartheta_1)(-t+u) + \\ 2y_2(u,\vartheta_1)(t-u) \end{pmatrix} du \\ \quad -\alpha t - \vartheta + 22t + 5, \\ y_2(\sigma_1^*) = \int_0^t \begin{pmatrix} 2x_2(u,\vartheta_1)(t-u) + \\ 3y_2(u,\vartheta_1)(-t+u) \end{pmatrix} du \\ \quad -\alpha t - \vartheta + 16t + 10, \end{cases} \tag{64}$$

$$\begin{cases} x_1(\sigma_2^*) = \int_0^t \begin{pmatrix} 3x_1(u,\vartheta_2)(-t+u) + \\ 2y_1(u,\vartheta_2)(t-u) \end{pmatrix} du \\ \quad -2\beta t - 3\beta + 21t + 4, \\ y_1(\sigma_2^*) = \int_0^t \begin{pmatrix} 2x_1(u,\vartheta_2)(t-u) + \\ 3y_1(u,\vartheta_2)(-t+u) \end{pmatrix} du \\ \quad -2\beta t - 2\beta + 15t + 9, \end{cases} \tag{65}$$

$$\begin{cases} x_2(\sigma_2^*) = \int_0^t \begin{pmatrix} 3x_2(u,\vartheta_2)(-t+u) + \\ 2y_2(u,\vartheta_2)(t-u) \end{pmatrix} du \\ \quad + 2\beta t + 2\beta + 21t + 4, \\ y_2(\sigma_2^*) = \int_0^t \begin{pmatrix} 2x_2(u,\vartheta_2)(t-u) + \\ 3y_2(u,\vartheta_2)(-t+u) \end{pmatrix} du + \\ \quad + 2t + 2\vartheta_2 + 15t + 9. \end{cases} \tag{66}$$

Now by using GMADM the solution of (62)−(66), can be expressed as;

$$\begin{cases} x_{1_0}(\sigma_1^*) = 2 + t(\vartheta_1 + 20) + 2\alpha, \\ y_{1_0}(\sigma_1^*) = 8 + t(\vartheta_1 + 14) + \vartheta, \\ x_{1_1}(\sigma_1^*) = -\frac{1}{6}t^3\vartheta_1 - \frac{16}{3}t^3 - 2\alpha t^2 + 5t^2, \\ y_{1_1}(\sigma_1^*) = \frac{1}{6}t^3\vartheta_1 + \frac{19}{3}t^3 + \frac{5}{2}t^2\vartheta_1 - 8t^2, \\ x_{1_{k+1}}(\sigma_1^*) = \int_0^t \begin{pmatrix} 3x_{1_k}(u,\vartheta_1)(-t+u) + \\ 2y_{1_k}(u,\vartheta_1)(t-u) \end{pmatrix} du, k \geqslant 1, \\ y_{1_{k+1}}(\sigma_1^*) = \int_0^t \begin{pmatrix} 2x_{1k}(u,\vartheta_1)(t-u) + \\ 3y_{1_k}(u,\vartheta_1)(-t+u) \end{pmatrix} du, k \geqslant 1. \end{cases} \tag{67}$$

$$\begin{cases} x_{2_0}(\sigma_1^*) = 5 + t(-\vartheta + 22) - \vartheta_1, \\ y_{2_0}(\sigma_1^*) = 10 + t(-\vartheta + 16) - \vartheta, \\ x_{2_1}(\sigma_1^*) = \frac{1}{6}t^3\vartheta_1 - \frac{17}{3}t^3 + \frac{1}{2}t^2\vartheta_1 + \frac{5}{2}t^2, \\ y_{2_1}(\sigma_1^*) = -\frac{1}{6}t^3\vartheta_1 + \frac{20}{3}t^3 - \frac{1}{2}t^2\vartheta_1 - 5t^2, \\ x_{2_{k+1}}(\sigma_1^*) = \int_0^t \begin{pmatrix} 3x_{2_k}(u,\vartheta_1)(-t+u) + \\ 2y_{2k}(u,\vartheta_1)(t-u) \end{pmatrix} du, k \geqslant 1, \\ y_{2_{k+1}}(\sigma_1^*) = \int_0^t \begin{pmatrix} 2x_{2_k}(u,\vartheta_1)(t-u) + \\ 3y_{2k}(u,\vartheta_1)(-t+u) \end{pmatrix} du, k \geqslant 1. \end{cases} \tag{68}$$

$$\begin{cases} x_{1_0}(\sigma_2^*) = 4 + t(-2\vartheta_2 + 21) - 3\vartheta_2, \\ y_{1_0}(\sigma_2^*) = 9 + t(-2\vartheta_2 + 15) - 2\vartheta_2, \\ x_{1_1}(\sigma_2^*) = \frac{1}{3}t^3\vartheta_2 - \frac{11}{2}t^3 + \frac{5}{2}t^2\vartheta_2 + 3t^2, \\ y_{1_1}(\sigma_2^*) = -\frac{1}{3}t^3\vartheta_2 + \frac{13}{2}t^3 - 3t^2\vartheta_2 - \frac{11}{2}t^2, \\ x_{1_{k+1}}(\sigma_2^*) = \int_0^t \begin{pmatrix} 2x_{1k}(u,\vartheta_2)(t-u) + \\ 3y_{1_k}(u,\vartheta_2)(-t+u) \end{pmatrix} du, k \geqslant 1, \\ y_{1_{k+1}}(\sigma_2^*) = \int_0^t \begin{pmatrix} 2x_{1_k}(u,\vartheta_2)(t-u) + \\ 3y_{1k}(u,\vartheta_2)(-t+u) \end{pmatrix} du, k \geqslant 1. \end{cases} \tag{69}$$

$$\begin{cases} x_{2_0}(\sigma_2^*) = 4 + t(2\vartheta_2 + 21) + 2\vartheta_2, \\ y_{2_0}(\sigma_2^*) = 9 + t(2\vartheta_2 + 15) + 2\vartheta_2, \\ x_{2_1}(\sigma_2^*) = -\frac{1}{3}t^3\vartheta_2 - \frac{11}{2}t^3 - t^2\vartheta_2 + 3t^2, \\ y_{2_1}(\sigma_2^*) = \frac{1}{3}t^3\vartheta_2 + \frac{13}{2}t^3 + t^2\vartheta_2 - \frac{11}{2}t^2, \\ x_{2_{k+1}}(\sigma_2^*) = \int_0^t \begin{pmatrix} 3x_{2_k}(u,\vartheta_2)(-t+u) + \\ 2y_{2k}(u,\vartheta_2)(t-u) \end{pmatrix} du, k \geqslant 1, \\ y_{2_{k+1}}(\sigma_2^*) = \int_0^t \begin{pmatrix} 2x_{2k}(u,\vartheta_2)(t-u) + \\ 3y_{2_k}(u,\vartheta_2)(-t+u) \end{pmatrix} du, k \geqslant 1. \end{cases} \tag{70}$$

By solving (66)−(70), we get the approximate solution after four iterations as follows:

$$
\begin{aligned}
&(x_1(\sigma_1^*), y_1(\sigma_1^*)) \\
&= \Bigg(2 + \vartheta_1 t + 20t + 2\alpha - \frac{1}{6}t^3\vartheta_1 - \\
&\frac{16}{3}t^3 - 2t^2\vartheta + 5t^2 + \frac{1}{24}t^5\vartheta_1 \\
&+ \frac{43}{30}t^5 + \frac{11}{12}t^4\vartheta_1 - \frac{31}{12}t^4 - \\
&\frac{29}{5040}t^7\vartheta_1 - \frac{25}{126}t^7 - \frac{8}{45}t^6\vartheta_1 \\
&+ \frac{181}{360}t^6 + \frac{169}{362880}t^9\vartheta_1 + \\
&\frac{1457}{90720}t^9 + \frac{373}{20160}t^8\vartheta_1 - \\
&\frac{211}{4032}t^8, 8 + \vartheta_1 t + 14t + \vartheta_1 + \\
&\frac{1}{6}t^3\vartheta_1 + \frac{19}{3}t^3 + \frac{5}{2}t^2\vartheta_1 - 8t^2 - \\
&\frac{7}{120}t^5\vartheta_1 - \frac{121}{60}t^5 - \frac{31}{24}t^4\vartheta_1 + \\
&\frac{11}{3}t^4 + \frac{41}{5040}t^7\vartheta_1 + \frac{101}{360}t^7 + \\
&\frac{181}{720}t^6\vartheta_1 - \frac{32}{45}t^6 - \frac{239}{362880}t^9\vartheta_1 \\
&- \frac{4121}{181440}t^9 - \frac{211}{8064}t^8\vartheta_1 + \\
&\frac{373}{5040}t^8\Bigg),
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
&(x_2(t,\vartheta_1), y_2(t,\vartheta_1)) \\
&= \Bigg(5 - \vartheta_1 t + 22t - \vartheta_1 + \frac{1}{6}t^3\vartheta_1 - \\
&\frac{17}{3}t^3 + \frac{1}{2}t^2\vartheta_1 + \frac{5}{2}t^2 - \\
&\frac{1}{24}t^5\vartheta_1 + \frac{91}{60}t^5 - \frac{5}{24}t^4\vartheta_1 - \\
&\frac{35}{24}t^4 + \frac{29}{5040}t^7\vartheta_1 - \frac{529}{2520}t^7 + \\
&\frac{29}{720}t^6\vartheta_1 + \frac{41}{144}t^6 - \frac{169}{362880}t^9\vartheta_1 \\
&+ \frac{3083}{181440}t^9 - \frac{169}{40320}t^8\vartheta_1 \\
&- \frac{239}{8064}t^8, 10 - \vartheta_1 t + 16t - \vartheta_1 \\
&- \frac{1}{6}t^3\vartheta_1 + \frac{20}{3}t^3 - \frac{1}{2}t^2\vartheta_1 - \\
&5t^2 + \frac{7}{120}t^5\vartheta_1 - \frac{32}{15}t^5 + \\
&\frac{7}{24}t^4\vartheta_1 + \frac{25}{12}t^4 - \frac{41}{5040}t^7\vartheta_1 + \\
&\frac{187}{630}t^7 - \frac{41}{720}t^6\vartheta_1 - \frac{29}{72}t^6 + \\
&\frac{239}{362880}t^9\vartheta_1 - \\
&\frac{109}{4536}t^9 + \frac{239}{40320}t^8\vartheta_1 + \frac{169}{4032}t^8\Bigg),
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
&(x_1(t,\vartheta_2), y_1(t,\vartheta_2)) \\
&= \Bigg(4 - 2\vartheta_2 t + 21t - 3\vartheta_2 + \frac{1}{3}t^3\vartheta_2 - \\
&\frac{11}{2}t^3 + \frac{5}{2}t^2\vartheta_2 + 3t^2 - \frac{1}{12}t^5\vartheta_2 \\
&+ \frac{59}{40}t^5 - \frac{9}{8}t^4\vartheta_2 - \frac{5}{3}t^4 + \frac{29}{2520}t^7\vartheta_2 - \\
&\frac{49}{240}t^7 + \frac{157}{720}t^6\vartheta_2 + \frac{13}{40}t^6 - \\
&\frac{169}{181440}t^9\vartheta_2 + \frac{1999}{120690}t^9 - \frac{61}{2688}t^8\vartheta_2 \\
&- \frac{341}{10080}t^8, 9 - 2\vartheta_2 t + 15t \\
&- 2\vartheta_2 - \frac{1}{3}t^3\vartheta_2 + \frac{13}{2}t^3 - 3t^2\vartheta_2 - \\
&\frac{11}{2}t^2 + \frac{7}{60}t^5\beta - \frac{83}{40}t^5 + \frac{19}{12}t^4\vartheta_2 \\
&+ \frac{19}{8}t^4 - \frac{41}{250}t^7\vartheta_2 + \frac{97}{336}t^7 - \frac{37}{120}t^6\vartheta_2 - \\
&\frac{331}{720}t^6 + \frac{239}{181440}t^9\vartheta_2 - \frac{2827}{120960}t^9 + \\
&\frac{647}{20160}t^8\vartheta_2 + \frac{643}{13440}t^8\Bigg),
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
&(x_2(\sigma_2^*), y_2(\sigma_2^*)) \\
&= \Bigg(4 + 2\beta t + 21 + 2\beta - \frac{1}{3}t^3\beta - \frac{11}{2}t^3 \\
&- t^2\beta + 3t^2 + \frac{1}{12}t^5\beta + \frac{59}{40}t^5 + \\
&\frac{5}{12}t^4\beta - \frac{5}{3}t^4 - \frac{29}{2520}t^7\beta - \\
&\frac{49}{240}t^7 - \frac{29}{360}t^6\beta + \frac{13}{40}t^6 + \\
&\frac{169}{20160}t^8\vartheta_2 - \frac{341}{10080}t^8, \\
&9 + 2\vartheta_2 t + 15t + 2\vartheta_2 + \frac{1}{3}t^3\vartheta_2 + \\
&\frac{13}{2}t^3 + t^2\vartheta_2 - \frac{11}{2}t^2 - \frac{7}{60}t^5\vartheta_2 \\
&- \frac{83}{40}t^5 - \frac{7}{12}t^4\vartheta_2 + \frac{19}{8}t^4 + \\
&\frac{41}{2520}t^7\vartheta_2 + \frac{97}{336}t^7 + \frac{41}{360}t^6\vartheta_2 - \\
&\frac{331}{720}t^6 - \frac{239}{181440}t^9\vartheta_2 - \frac{2827}{120960}t^9 \\
&- \frac{239}{20160}t^8\vartheta_2 + \frac{643}{13440}t^8\Bigg).
\end{aligned} \tag{74}
$$

In Table 6, $x_1(\sigma_1^*)$,$x_2(\sigma_1^*)$, $y_1(\sigma_1^*)$, and $y_2(\sigma_1^*)$ represents analytical solution of the membership functions of Example 2 for $\vartheta_1 \in 0, 1.0]$.

In Table 7,$x_1(\sigma_2^*)$, $x_2(\sigma_2^*)$, $y_1(\sigma_2^*)$, and $y_2(\sigma_2^*)$ represents analytical solution of non-membership function of Example 2 for $\vartheta_2 \in [0, 1.0]$.

Table 6. Analytical solution of the membership functions of the Example 2 for θ_1 belong to [0, 1.0] after 2 iterations.

ϑ_1	$x_1(t,\vartheta_1)$	$y_1(t,\vartheta_1)$	$x_2(t,\vartheta_1)$	$y_2(t,\vartheta_1)$
0	12.47	21.60	15.88	27.53
0.1	12.68	21.96	15.74	27.29
0.2	12.88	22.31	15.60	27.05
0.3	13.08	22.67	15.47	26.82
0.4	13.27	23.02	15.33	26.58
0.5	13.49	23.38	15.19	26.34
0.6	13.69	23.73	15.05	26.10
0.7	13.90	24.09	15.92	25.87
0.8	14.10	24.44	14.78	25.63
0.9	14.30	24.80	14.64	25.39
1.0	14.51	25.15	14.51	25.15

Table 7. Analytical solution of the non-membership functions of the Example 2 for θ_1 belong to [0, 1.0] after 2 iterations.

ϑ_2	$x_1(t,\vartheta_2)$	$y_1(t,\vartheta_2)$	$x_2(t,\vartheta_2)$	$y_2(t,\vartheta_2)$
0	14.51	25.15	14.51	25.15
0.1	14.17	24.57	14.78	25.63
0.2	13.83	23.97	15.05	26.10
0.3	13.49	23.38	15.33	26.58
0.4	13.15	22.78	15.60	27.05
0.5	12.81	22.19	15.88	27.53
0.6	12.47	21.60	16.15	28.00
0.7	12.13	21.01	16.42	28.48
0.8	11.79	20.41	16.70	28.95
0.9	11.45	19.82	16.97	29.43
1.0	11.10	19.23	17.24	29.90

The exact solution of Example 2 by classical method is given as follows:

$$(x_1(\sigma_1^*), y_1(\sigma_1^*)) = \Bigg\{ -\frac{3}{5}sin(\sqrt{5t})\sqrt{5} + \left(-9 + \frac{1}{2}\vartheta_1\right)cos\left(\sqrt{5t}\right) + (11 + \vartheta_1)sin(t) + \left(11 + \frac{3}{2}\vartheta_1\right)cos(t), \quad \frac{3}{5}sin\left(\sqrt{5t}\right)\sqrt{5} - \left(-9 + \frac{1}{2}\vartheta_1\right)cos\left(\sqrt{5t}\right) + (11 + \vartheta_1)sin(t) + \left(11 + \frac{3}{2}\vartheta_1\right)cos(t)\Bigg\}, \tag{75}$$

$$(x_2(\sigma_1^*), y_2(\sigma_1^*)) = \Bigg\{ -\frac{3}{5}sin(\sqrt{5t})\sqrt{5} - \frac{17}{2}cos\left(\sqrt{5t}\right) + (13 - \vartheta_1)sin(t) + \left(\frac{27}{2} - \vartheta_1\right)cos(t), \quad \frac{3}{5}sin\left(\sqrt{5t}\right)\sqrt{5} + \frac{17}{2}cos\left(\sqrt{5t}\right) + (13 - \vartheta_1)sin(t) + \left(\frac{27}{2} - \vartheta_1\right)cos(t)\Bigg\}, \tag{76}$$

Table 8. The exact solution to the membership functions for the $\theta = 1$that belong to [0, 1.0] in Example 2.

ϑ_1	$x_1\ (t,\vartheta_1)$	$y_1\ (t,\vartheta_1)$	$x_2\ (t,\vartheta_1)$	$y_2\ (t,\vartheta_1)$
0	9.782	20.07	13.16	23.01
0.1	9.983	20.22	13.02	22.87
0.2	10.18	20.39	12.88	22.73
0.3	10.39	20.55	12.74	22.59
0.4	10.59	20.70	12.61	22.47
0.5	10.79	20.86	12.48	22.33
0.6	10.99	21.01	12.34	22.19
0.7	11.20	21.18	12.20	22.05
0.8	11.40	21.34	12.07	21.93
0.9	11.60	21.49	11.94	21.79
1.0	11.80	21.65	11.80	21.65

$$(x_1(\sigma_2^*), y_1(\sigma_2^*)) = \Bigg\{(12 - 2\vartheta_2)sin(t) + \left(\frac{25}{2} - \frac{5}{2}\vartheta_2\right)cos(t) - \frac{3}{5}sin\left(\sqrt{5t}\right)\sqrt{5} + \left(-\frac{17}{2} - \frac{1}{2}\vartheta_2\right)cos\left(\sqrt{5t}\right), \quad (12 - 2\vartheta_2)sin(t) + \left(\frac{25}{2} - \frac{5}{2}\vartheta_2\right)cos(t) + \frac{3}{5}sin\left(\sqrt{5t}\right)\sqrt{5} - \left(-\frac{17}{2} - \frac{1}{2}\vartheta_2\right)cos\left(\sqrt{5t}\right)\Bigg\}, \tag{77}$$

$$(x_2(\sigma_2^*), y_2(\sigma_2^*)) = \Bigg\{ -\frac{3}{5}sin\left(\sqrt{5t}\right)\sqrt{5} - \frac{17}{2}cos\left(\sqrt{5t}\right) + (12 + 2\vartheta_2)sin(t) + \left(\frac{25}{2} + 2\vartheta_2\right)cos(t), \quad \frac{3}{5}sin\left(\sqrt{5t}\right)\sqrt{5} + \frac{17}{2}cos\left(\sqrt{5t}\right) + (12 + 2\vartheta_2)sin(t) + \left(\frac{25}{2} + 2\vartheta_2\right)cos(t)\Bigg\}. \tag{78}$$

In Table 8, $x_1(\sigma_1^*)$,$x_2(\sigma_1^*), y_1(\sigma_1^*)$, and $y_2(\sigma_1^*)$ represents analytical solution of the membership functions of Example 2 for $\vartheta_1 \in 0, 1.0]$

In Table 9,$x_1(\sigma_2^*)$, $x_2(\sigma_2^*), y_1(\sigma_2^*)$, and $y_2(\sigma_2^*)$ represents analytical solution of non-membership function of Example 2 for $\vartheta_2 \in [0, 1.0]$.

Table 10. shows numerical comparison of computational time seconds (CPU), number of iterations (Ns), and residual error (Err) of GMADM, GTM, and GDM respectively for solving system of intuitionistic fuzzy initial value problems used in Example 2.

Advantages of the GMADM

- GMADM is found to converge very quickly and to be more accurate than GTM and GDM in solving a system of fuzzy intuitionistic differential equations.

Table 9. The exact solution to the non-membership functions for the $\theta = 1$ that belong to [0, 1] in Example 2.

ϑ_2	x_1 (t,ϑ_2)	y_1 (t,ϑ_2)	x_2 (t,ϑ_2)	y_2 (t,ϑ_2)
0	11.79	21.65	11.80	21.65
0.1	11.46	21.36	12.07	21.93
0.2	11.12	21.06	12.34	22.19
0.3	10.79	20.77	12.61	22.47
0.4	10.44	20.48	12.88	22.73
0.5	10.11	20.19	13.16	23.01
0.6	9.770	19.89	13.42	23.27
0.7	9.438	19.60	13.70	23.55
0.8	9.096	19.30	13.96	23.82
0.9	8.754	19.01	14.24	24.09
1.0	8.422	18.72	14.52	24.37

- The main advantage of the GMADM method is that it can solve all types of fuzzy differential equations and system of intuitionistic with more generalized fuzzy numbers.

Table 10. A numerical comparison of GMADM, GTM, and GDM for solving the intuitionistic fuzzy system of differential equations used in Example 2.

Parameter	*GMADM*	GTM	GDM
CPU	0.0013	2.4043	0.0624
Ns	04	04	05
Err	$9.8e^{-15}$	$3.4e^{-3}$	$0.6e^{-7}$

- The GMADM also has the practical advantage of reducing computing cost while preserving improved accuracy of the numerical solution.
- GMADM can efficiently, rapidly, and accurately solve a large class system of fuzzy intuitionistic differential equations with closed form solutions that rapidly converge to exact solutions.
- The GMADM has proved to be very efficient and yields significant accuracy and computation time savings, as illustrated in Figures 1 to 3 and Tables 1 to 10.

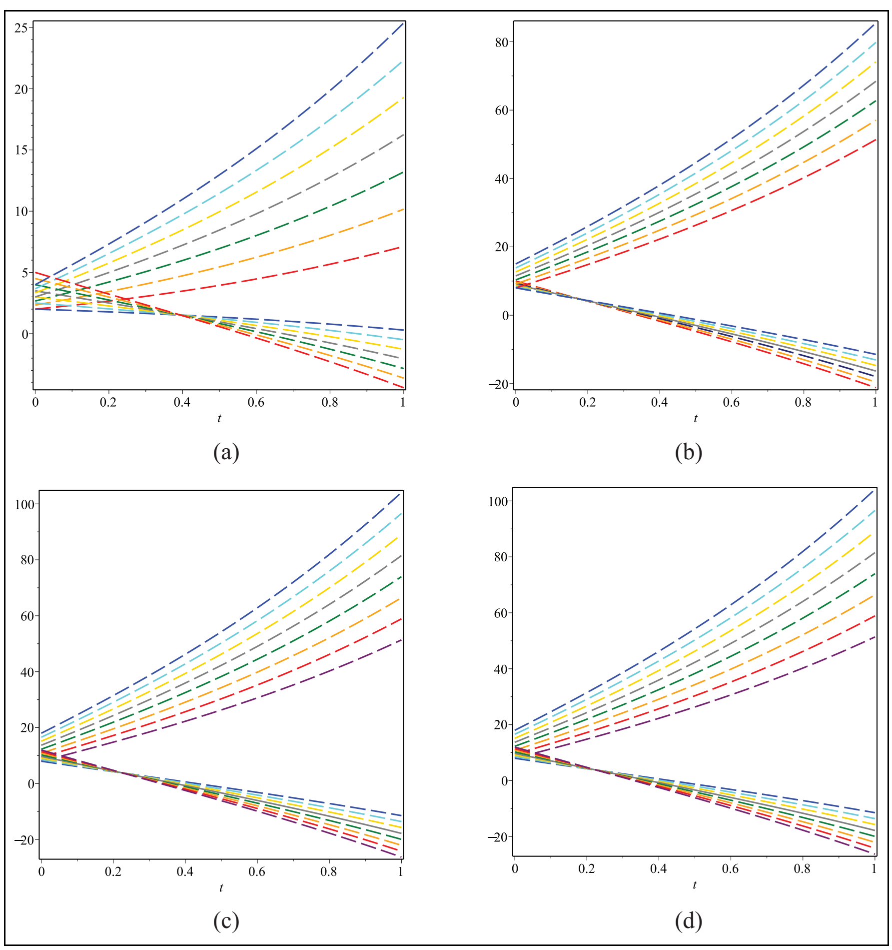

Figure 3. (a–d): clearly shows that numerical approximate solution obtained by HPM are exactly matched with analytical solution of system of intuitionistic triangular fuzzy initial value problems used in Example 2. (a) Numerical and analytical solution of x_1, (b) numerical and analytical solution of x_2, (c) numerical and analytical solution of x_3, (d) numerical and analytical solution of TIFLSEs used in Example 2.

Conclusion

In this work, Generalized Modified Adomian Decomposition Method have been utilized for computing the approximate solution of the linear system of intuitionistic fuzzy initial value problems. We used the initial conditions as a triangular intuitionistic fuzzy numbers. We have applied this procedure to mechanical engineering problems. From all Tables 1 to 10 and Figures 1 to 3, clearly shows the dominance efficiency of GMADM over GTM and GDM in terms of computational time, number of iterations and in errors. Moreover, by comparing the approximate results with exact solution, we have shown that this method is more reliable. Future studies will therefore focus on the solution of systems of higher order generalized triangular intuitionistic fuzzy differential equations as well as a system of nonlinear first order differential equations and their application in a generalized intuitionistic fuzzy environment utilizing GMADM.

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Authors' contributions

All authors contributed equally to the preparation of this manuscript.

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Data availability

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