Introducing Interpolative Boolean Algebra for Interval-Valued Fuzzy Sets

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Abstract. Interval-valued fuzzy sets (IVFSs) are an extension of traditional fuzzy sets by representing membership degrees as intervals rather than crisp numbers. It enables a more sophisticated modeling of imprecision. Despite these advantages, there remains substantial room for improvement in ensuring logical consistency and in adequately capturing dependencies between interval bounds. In this paper, we propose applying Interpolative Boolean algebra (IBA) as an algebraic framework for IVFS theory. Specifically, we define conjunction and disjunction operations for IVFS that are directly derived from IBA. These logical operations are based on the generalized product, an operator that can be realized by any t-norm whose result is greater than or equal to the minimum t-norm and less than or equal to the Łukasiewicz t-norm. Finally, we investigate the validity of the laws of contradiction and excluded middle within the proposed approach.

Keywords: Interpolative Boolean Algebra · Interval-Valued Fuzzy Sets · Boolean Laws.

1 Introduction

Interval-valued fuzzy sets (IVFSs) are one of the most prominent extensions of traditional fuzzy sets [1, 2]. This approach is widely used in decision-making, data analysis, and other areas where modeling uncertainty and imprecision of belonging to the concept is of critical importance.

IVFSs have a strong mathematical foundation (see, e.g., [3]). They differ from the broader concept of type 2 fuzzy sets [5] in the precision of their membership

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grades; IVFS uses a fixed interval, while type 2 fuzzy sets use a fuzzy set as the membership grade. As in most fuzzy logic-based theories, IFVS is not consistent with the Boolean framework in a general case, i.e., the laws of excluded middle and contradiction are not satisfied for any involutive interval-valued negation [4]. On the one hand, this may be regarded as a well-known fact that usually does not hinder practical application; on the other hand, it presents an opportunity for further research. One possible research direction is the incorporation of real-valued Boolean-consistent algebras, e.g., interpolative Boolean algebra (IBA) [6], into IVFS theory.

In this paper, we explore the potential benefits of applying IBA as an algebraic framework to deal with IVFS values. Specifically, we introduce an IBA-based generalization of conjunction and disjunction for IVFS, based on suitably generalized Boolean polynomials. Furthermore, we extend the list of IBA transformation rules to better accommodate the nature of IVFSs. The mathematical properties of the resulting IBA-IVFS approach are proposed to demonstrate their generality.

The remainder of this paper is organized as follows. Section 2 outlines the theoretical background of IVFSs and IBA. Section 3 introduces the IBA-IVFS approach and the analysis of the laws of excluded middle and contradiction within the proposed approach. Finally, Section 4 concludes the article.

2 Theoretical Background

This section recalls the preliminaries of IVFS and IBA which are used throughout the article.

2.1 Interval-Valued Fuzzy Sets

In IVFS theory, any entity can be represented by two membership functions defining the lower and upper bounds of membership. Formally, IVFS membership for an element in the referential set is given as $x = [\underline{x}, \overline{x}]$, where $(\underline{x}, \overline{x}) \in [0,1]^2$ and $\underline{x} \leq \overline{x}$ [3]. The gap between the two memberships quantifies the degree of vagueness or uncertainty, i.e. $\overline{x} - \underline{x} = W_x$.

Logical connectives in IVFS theory are first generated on the basis on standard fuzzy t-norms and t-conorms, and then generalized as interval valued t-norms and s-norms [3, 4]. As a negation operator, standard interval valued negation is a common choice [3]. A subclass of s-norms that are s-representable in IVFS do not satisfy the LEM, and a subclass of t-norms that are t-representable in IVFS do not satisfy the LC, for any involutive IV negation [4].

2.2 Interpolative Boolean Algebra

IBA is a real-valued realization of Boolean algebra [6] and serves as a basis for Boolean consistent fuzzy logic [7], intuitionistic fuzzy calculus that adheres to the foundations of intuitionism [8], and other approaches based on generalized fuzzy sets [9]. From a technical point of view, IBA consists of two levels: symbolic and valued.

On the IBA symbolic level, any logical function φ can and should be calculated directly on the basis of the structure of its components and transformed into generalized Boolean polynomial (GBP) $\varphi^{\otimes}$. The IBA transformation rules for both complex logical expression and individual attributes can be found in [6], while the automated procedure is presented in [10]. Transformation rules are value-independent, while GBP is a polynomial with standard addition, standard subtraction, and the generalized product $\otimes$ (GP) as operators.

After applying the transformation rules, the valued level of IBA is introduced. At this level, values are assigned to each attribute and an appropriate realization of GP is selected. The GP is a subclass of t-norms that ranges between the minimum and the Łukasiewicz t-norm and is usually selected according to the degree of correlation in the data. The choice of appropriate operator for GP in various real-world situations is thoroughly studied in literature [11].

3 Interpolative Boolean Algebra for Interval-Valued Fuzzy Sets

In this section, we propose an IBA-based approach to IVFS theory. In the IBA-IVFS approach, we aim to leverage the strengths of both frameworks. In fact, we maintain the modeling capabilities and applicability of IVFS, while the mathematical properties, transformation procedures, and generality of the approach are defined similarly to those of IBA.

That is, the logical operations of conjunction and disjunction are defined as a direct extension of the IBA transformation rules. On the other hand, we have decided to maintain the standard IVFS negation.

$$(x \wedge y)^{\otimes} = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}] \tag{1}$$

$$(x \vee y)^{\otimes} = [\underline{x} + \underline{y} - \underline{x} \otimes \underline{y},\ \overline{x} + \overline{y} - \overline{x} \otimes \overline{y}] \tag{2}$$

$$(\neg x)^{\otimes} = [1 - \overline{x},\ 1 - \underline{x}] \tag{3}$$

Since the negation operator may lead to interaction between the lower and upper membership function values, it is necessary to introduce an additional transformation rule to handle this situation.

$$\underline{x} \otimes \overline{x} = \underline{x} \tag{4}$$

The rationale behind this transformation rule is that the conjunction of the lower and upper membership function values should be equal to the lower membership. In contrast to traditional fuzzy and interval-valued fuzzy operators, which are t-norm dependent and for which this relation holds only in marginal cases, the operator considered here is independent and can be treated as an axiom. Consequently, at the symbolic level of IBA, this rule plays a crucial role in ensuring the logical consistency of the proposed approach. Finally, although this rule is understandable and reasonable on an intuitive level, it has a strong foundation based on IBA theory. Namely, since these memberships represent the same attribute (i.e., the same fuzzy variable or set), the min operator is a clear and natural choice.

On the other hand, the realization of GP at the value level enables the adequate capture of dependencies between interval bounds. GP can be implemented using a wide range of t-norms, allowing it to be tailored to the preferences and requirements of the decision-maker.

3.1 Mathematical properties of IBA-IVFS approach

The laws of commutativity and associativity hold in this approach due to directly transferred properties of GP. Further, negation is involutive since the negation operator is directly taken from IVFS. Finally, we explore the validity of the laws of excluded middle and contradiction within the proposed IBA-IVFS approach.

Law of excluded middle

$$(x \vee \neg x)^{\otimes} = \left[\underline{x} + (1-\overline{x}) - \underline{x} \otimes (1-\overline{x}),\ \overline{x} + (1-\underline{x}) - \overline{x} \otimes (1-\underline{x})\right] \tag{5}$$

$$(x \vee \neg x)^{\otimes} = \left[\underline{x} + 1 - \overline{x} - \underline{x} + \underline{x} \otimes \overline{x},\ \overline{x} + 1 - \underline{x} - \overline{x} + \overline{x} \otimes \underline{x}\right] \tag{6}$$

$$(x \vee \neg x)^{\otimes} = \left[1 - W_x,\ 1\right] \tag{7}$$

Law of contradiction

$$(x \wedge \neg x)^{\otimes} = \left[\underline{x} \otimes (1-\overline{x}),\ \overline{x} \otimes (1-\underline{x})\right] \tag{8}$$

$$(x \wedge \neg x)^{\otimes} = \left[\underline{x} - \underline{x} \otimes \overline{x},\ \overline{x} - \overline{x} \otimes \underline{x}\right] \tag{9}$$

$$(x \wedge \neg x)^{\otimes} = \left[0,\ W_x\right] \tag{10}$$

Clearly, both laws are not satisfied in the traditional sense, since these expressions are not equal to the crisp neutral elements [0,0] and [1,1]. Nevertheless, they produce a constant result that retains the IVFS nature, i.e. it incorporates a fixed uncertainty inherent to the input IVFS. Additionally, the results are not value-dependent and can be considered as an identity. Consequently, the presented results provide a basis for further research, both in theory, particularly with respect to ensuring the validity of these laws, and in practice, through the application of IBA-IVFSs to evaluation and ranking problems involving negation in logical aggregation functions.

4 Conclusion and future work

The proposed IBA-IVFS approach represents an initial attempt to introduce IBA-based operations within the IVFS framework. The logical operations of conjunction and disjunction are defined using the generalized product, which offers a wide range of modeling possibilities. The laws of excluded middle and contradiction are analyzed within this approach, showing that they are not satisfied in the traditional crisp sense. Instead, their outcomes hold as identities within the IVFS framework, fully consistent with the interval-valued fuzzy nature.

Future work can proceed in two main directions. First, the mathematical properties of the IBA-IVFS approach should be studied in detail, with particular attention to De Morgan's laws and neutral elements. Second, the proposed approach can be applied to ranking and selection tasks. One possible area of application is education and grading, since grading systems are closely related to the concept of interval-valued fuzzy values. Another possible area is the evaluation of companies by economic, social and governmental aspects in an environment of a higher level of uncertainty.

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