

Available at:

www.pmf.ni.ac.yu/sajt/publikacije/publikacije_pocetna.html

Filomat **20:2** (2006), 101–105

A LITTLEWOOD-PALEY TYPE INEQUALITY FOR HARMONIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{R}^N$

OLIVERA DJORDJEVIĆ

ABSTRACT. It is proved the following: If u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$, and $0 < p < 1$, then there holds the inequality

$$\sup_{0 < r < 1} \int_{\partial B} |u(ry)|^p d\sigma \leq |u(0)|^p + C_{p,N} \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x).$$

In the case $p > (N - 2)/(N - 1)$, this was proved by Stević [17].

Let $\mathbb{R}^N$ ($N \geq 2$) denote the N -dimensional Euclidean space. In [17], Stević proved that if u is a function harmonic in the unit ball $B \subset \mathbb{R}^N$, and $\frac{N-2}{N-1} \leq p < 1$, then there holds the inequality

$$(1) \quad \sup_{0 < r < 1} M_p^p(r, u) \leq C_1 |u(0)|^p + C_2 \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x).$$

Here dV denotes the Lebesgue measure in $\mathbb{R}^N$ normalized so that $V(B) = 1$, and as usual

$$M_p^p(r, u) = \int_{\partial B} |u(ry)|^p d\sigma,$$

where $d\sigma$ is the normalized surface measure on the sphere ∂B . It is the aim of this note to remove the strange condition $(N - 2)/(N - 1) \leq p < 1$. This condition appears in [17] because the proof in the paper is based on the fact, due Stein and Weiss [16, 15], that $|\nabla u|^p$ is subharmonic for $p \geq (N - 2)/(N - 1)$. Our result is slightly stronger than (1):

¹Received: October 2, 2006

2000 *Mathematics Subject Classification.* 31B05.

Key words and phrases. Littlewood-Paley inequalities, harmonic functions.

Supported by MNZŽS, Project 144010.

Theorem 1. *If u is a function harmonic in B , and $0 < p < 1$, then there holds the inequality*

$$(2) \quad \sup_{0 < r < 1} M_p^p(r, u) \leq |u(0)|^p + C \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x),$$

where C is a constant depending only on p and N .

In the case $N = 2$, this theorem was proved by Flett [2]. Inequality (2) holds for $1 < p < 2$ as well, while if $p > 2$, then there holds the reverse inequality; these inequalities are due to Littlewood and Paley [6]. Elementary proofs of the Littlewood-Paley inequalities are given in [12] and [7, 14] ($p > 2$).

Observe that if $u > 0$ in B , and $0 < p < 1$, then (2) is completely trivial because then function u^p is superharmonic and therefore

$$\sup_{0 < r < 1} M_p^p(r, u) \leq |u(0)|^p.$$

Thus (2) shows in particular how much $|u|^p$ is far from being superharmonic.

Our proof of Theorem 1 is based on a fundamental result of Hardy and Littlewood [3] and Fefferman and Stein [1] on subharmonic behavior of $|u|^p$. We state this result in the following way.

Lemma 1. *If $U \geq 0$ is a function subharmonic in $B(a, 2\varepsilon)$ ($a \in \mathbb{R}^N$, $\varepsilon > 0$), then there holds the inequality*

$$(3) \quad \sup_{x \in B(a, \varepsilon)} U(x)^p \leq C \varepsilon^{-N} \int_{B(a, 2\varepsilon)} U^p dV, \quad 0 < \varepsilon < 1,$$

where C depends only on p, N .

Here $B(a, r)$ denotes the ball of radius r centered at a . For simple proofs of Lemma 1 we refer to [9, 13], and for generalizations to various classes of functions, we refer to [4, 5, 8, 10, 11]. From Lemma 1 we shall deduce the following crucial fact:

Lemma 2. *Let $r_j = 1 - 2^{-j}$ for $j \geq 0$, and $r_{-1} = 0$. If $0 < p < 1$ and u is harmonic in B , then there holds inequality*

$$M_p^p(r_{j+1}, u) - M_p^p(r_j, u) \leq C \int_{r_{j-1} \leq |x| \leq r_{j+2}} (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x), \quad j \geq 0,$$

where C depends only on p and N .

Proof. We start from the inequality

$$(4) \quad M_p^p(r_{j+1}, u) - M_p^p(r_j, u) \leq \int_S |u(r_{j+1}y) - u(r_j y)|^p d\sigma(y).$$

By Lagrange's theorem,

$$(5) \quad |u(r_{j+1}y) - u(r_jy)| \leq (r_{j+1} - r_j) \sup_{r_j < r < r_{j+1}} |\nabla u(r y)| \leq 2^{-j} \sup_{r_j < r < r_{j+1}} |\nabla u(r y)|.$$

Hence, by Lemma 1 with $U = |\nabla u|$, $a = a_j = (r_j + r_{j+1})y/2$ and $\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2}$,

$$(6) \quad |u(r_{j+1}y) - u(r_jy)|^p \leq C 2^{-jp} 2^{jN} \int_{B(a_j, 2^{-j-1})} |\nabla u(x)|^p dV(x).$$

On the other hand, simple calculation shows that $|x - a_j y| \leq 2^{-j-1}$ implies

$$2^{-j-2} \leq 1 - |x|, \quad |x - y| \leq 2^{-j+1}.$$

Hence

$$2^{-j} 2^{jN} \leq 2^{N+2} P(x, y), \quad \text{for } x \in B(a_j, 2^{-j-1}),$$

where P denotes the Poisson kernel,

$$(7) \quad P(x, y) = \frac{1 - |x|^2}{|x - y|^N}.$$

From this and (6) we get

$$(8) \quad |u(r_{j+1}y) - u(r_jy)|^p \leq C 2^{-j(p-1)} \int_{r_{j-1} \leq |x| \leq r_{j+2}} P(x, y) |\nabla u(x)|^p dV(x),$$

where we have used the inclusion

$$\{x : |x - a_j| \leq 2^{-j-1}\} \subset \{x : r_{j-1} \leq |x| \leq r_{j+2}\}.$$

Now we integrate (8) over ∂B and use the formula

$$\int_S P(x, y) d\sigma(y) = 1$$

to get

$$\begin{aligned} \int_S |u(r_{j+1}y) - u(r_jy)|^p d\sigma(y) &\leq C 2^{-j(p-1)} \int_{r_{j-1} \leq |x| \leq r_{j+2}} |\nabla u(x)|^p dV(x) \\ &\leq C \int_{r_{j-1} \leq |x| \leq r_{j+2}} (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x). \end{aligned}$$

Combining this with (4) we get the desired result. $\square$

Proof of Theorem 1. Let $n \geq 1$. By Lemma 2, we have

$$\begin{aligned}
M_p^p(r_n, u) - |u(0)|^p &= M_p^p(r_n, u) - M_p^p(r_0, u) \\
&= \sum_{j=0}^{n-1} M_p^p(r_{j+1}, u) - M_p^p(r_j, u) \\
&\leq C \sum_{j=0}^{n-1} \int_{r_{j-1} \leq |x| \leq r_{j+2}} (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x) \\
&\leq 3C \int_{|x| \leq r_{n+1}} (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x) \\
&\leq 3C \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x).
\end{aligned}$$

This proves the inequality

$$(9) \quad M_p^p(r, u) \leq |u(0)|^p + C \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x),$$

for $r = r_n$. If $r \in (0, 1)$ is arbitrary, we choose n so that $r_n \leq r \leq r_{n+1}$. Then we have

$$|u(ry) - u(r_n y)| \leq 2^{-n} \sup_{r_n < r < r_{n+1}} |\nabla u(ry)|.$$

Hence, by the proof of Lemma 2,

$$\begin{aligned}
M_p^p(r, u) - M_p^p(r_n, u) &\leq C \int_{r_{n-1} \leq |x| \leq r_{n+2}} (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x) \\
&\leq C \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p dV(x).
\end{aligned}$$

This completes the proof.

Acknowledgement. The author wishes to thank Professor Pavlović for introducing into the subject and for valuable comments.

REFERENCES

- [1] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137 - 193.
- [2] T. M. Flett, *Mean values of power series*, Pacific J. Math. 25(1968), 463-494.
- [3] G.H. Hardy and J.E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math. **167** (1932), 405-423.
- [4] M. Jevtić, M. Pavlović, *M-Besov p -classes and Hankel operators in the Bergman spaces on the unit ball*, Arch. Math. 61(1993), 367-376.
- [5] M. Jevtić, M. Pavlović, *On subharmonic behaviour of functions and their tangential derivatives on the unit ball in $\mathbb{C}^n$* , Indian J. Pure Appl. Math. 30(1999), no. 4, 407-418.
- [6] J.E. Littlewood and R.E.A.C. Paley, *Theorems on Fourier series and power series. II*, Proc. Lond. Math. Soc. 42(1936), 52-89.

- [7] D.H. Luecking, *A new proof of an inequality of Littlewood and Paley*, Proc. Amer. Math. Soc. 103(1988), 887-893.
- [8] M. Pavlović, *Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball*, Indag. Math. (N.S.) 2(1991), no. 1, 89-98.
- [9] M. Pavlović, *On subharmonic behaviour and oscillation of functions on balls in $\mathbf{R}^n$* , Publ. Inst. Math. (Beograd) (N.S.) **55(69)** (1994), 18-22.
- [10] M. Pavlović, *Subharmonic behaviour of smooth functions*, Mat. Vesnik **48** (1996), no. 1-2, 15-21.
- [11] M. Pavlović, *Decompositions of L_p and Hardy spaces of polyharmonic functions*, J. Math. Anal. Appl. **216**(1997), 499-509.
- [12] M. Pavlović, *A Littlewood-Paley theorem for subharmonic functions with subharmonic Laplacian*, Publ. Inst. Math. (Belgrade) 68(82)(2000), 77-82.
- [13] M. Pavlović, *Introduction to function spaces on the disk*, Posebna izdanja 20, Matematički Institut SANU, 2004.
- [14] M. Pavlović, *A short proof of an inequality of Littlewood and Paley*, Proc. Am. Math. Soc. to appear.
- [15] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
- [16] E. M. Stein, G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of H^p - spaces*, Acta Math. 103 (1960) 25-62.
- [17] S. Stević, *A Littlewood-Paley type inequality*, Bull. Braz. Math. Soc. (N.S.) **34**(2003), no. 2, 211-217.

FAKULTET ORGANIZACIONIH NAUKA, JOVE ILIĆA 154, BELGRADE, SERBIA
E-mail address: oliveradj@fon.bg.ac.yu