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Note

Subharmonicity of $|f|^p$ for quasiregular harmonic functions, with applications

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Abstract

We prove that if f is a quasiregular harmonic function, then there exists a number $q \in (0,1)$ such that $|f|^q$ is subharmonic, and use this fact to generalize a result of Rubel, Shields, and Taylor, and Tamrazov, on the moduli of continuity of holomorphic functions.

Keywords: Subharmonic functions; Quasiregular functions; Moduli of continuity

It is well known that if f is a complex-valued harmonic function defined in a region G of the complex plane $\mathbb{C}$, then $|f|^p$ is subharmonic for $p \geqslant 1$, and that in the general case is not subharmonic for $p < 1$. However, if f is holomorphic, then $|f|^p$ is subharmonic for every $p > 0$. In this paper we consider k-quasiregular harmonic functions $(0 < k < 1)$. We recall that a harmonic function is k-quasiregular if

$$\left|\bar\partial f(z)\right| \leqslant k\left|\partial f(z)\right|, \quad z \in G,$$

where

$$\bar\partial f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \partial f(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad z = x + iy.$$

We prove that $|f|^p$ is subharmonic for $p \geqslant 4k/(1+k)^2 =: q$ as well as that the exponent q (< 1) is the best possible (see Theorem 1). The fact that $q < 1$ enables us to prove that if f is quasiregular in the unit disk $\mathbb{D}$ and continuous on $\overline{D}$, then $\tilde\omega(f,\delta) \leqslant \text{const.}\,\omega(f,\delta)$, where $\tilde\omega(f,\delta)$ (respectively $\omega(f,\delta)$) denotes the modulus of continuity of f on $\mathbb{D}$ (respectively $\partial\mathbb{D}$); see Theorem 2. In the case $k = 0$ (when f is holomorphic) this fact is known and was proved by Rubel, Shields, and Taylor [2], and Tamrazov [3].

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1. Subharmonicity of $|f|^p$

Theorem 1. *If f is a complex-valued k-quasiregular harmonic function defined on a region $G \subset \mathbb{C}$, and $q = 4k/(k+1)^2$, then $|f|^q$ is subharmonic. The exponent q is optimal.*

Recall that a continuous function u defined on a region $G \subset \mathbb{C}$ is subharmonic if for all $z_0 \in G$ there exists $\varepsilon > 0$ such that

$$u(z_0) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u\big(z_0 + re^{it}\big)\, dt, \quad 0 < r < \varepsilon. \tag{1}$$

If $u(z_0) = |f(z_0)|^2 = 0$, then (1) holds. If $u(z_0) > 0$, then there exists a neighborhood U of z_0 such that u is of class $C^2(U)$ (because the zeroes of u are isolated), and then we may prove that $\Delta u \geqslant 0 \in U$. Thus the proof reduces to proving that $\Delta u(z) \geqslant 0$ whenever $u(z) > 0$. In order to do this we will calculate Δu. Before that, we state some lemmas. The next two lemmas are well known and easy to prove.

Lemma 1. *If $u > 0$ is a C^2 function defined on a region in $\mathbb{C}$, and $\alpha \in \mathbb{R}$, then*

$$\Delta\big(u^\alpha\big) = \alpha u^{\alpha-1} \Delta u + \alpha(\alpha - 1) u^{\alpha-2} |\nabla u|^2. \tag{2}$$

Lemma 2. *If $u > 0$ is a C^2 function defined on a region in $\mathbb{C}$, then*

$$|\nabla u|^2 = 4|\partial u|^2 \quad \textit{and} \quad \Delta u = 4\partial\bar{\partial} u. \tag{3}$$

Lemma 3. *If $f = g + \bar{h}$, where g and h are holomorphic functions, then*

$$\Delta\big(|f|^2\big) = 4\big(|g'|^2 + |h'|^2\big). \tag{4}$$

Proof. Since $|f|^2 = (g + \bar{h})(\bar{g} + h)$, we have

$$\begin{aligned}\Delta\big(|f|^2\big) &= 4\partial\big(\overline{h'}(\bar{g} + h) + (g + \bar{h})\overline{g'}\big)\\ &= 4(\overline{h'}h + g\overline{g'})\\ &= 4\big(|g'|^2 + |h'|^2\big). \qquad \square\end{aligned}$$

Lemma 4. *If $f = g + \bar{h}$, where g and h are holomorphic functions, then*

$$\big|\nabla\big(|f|^2\big)\big|^2 = 4\big(|g'|^2 + |h'|^2\big)|f|^2 + 8\,\mathrm{Re}\big(\overline{g'}h' f^2\big). \tag{5}$$

Proof. We have

$$\begin{aligned}\big|\nabla\big(|f|^2\big)\big|^2 &= 4\big|\partial\big(|f|^2\big)\big|^2\\ &= 4\big|\partial\big((g + \bar{h})(\bar{g} + h)\big)\big|^2\\ &= 4|g'\bar{f} + fh'|^2\\ &= 4\big(|g'|^2 + |h'|^2\big)|f|^2 + 8\,\mathrm{Re}\big(\overline{g'}h' f^2\big). \qquad \square\end{aligned}$$

Lemma 5. *If $f = g + \bar{h}$, where g and h are holomorphic functions, then*

$$\Delta\big(|f|^p\big) = p^2\big(|g'|^2 + |h'|^2\big)|f|^{p-2} + 2p(p-2)|f|^{p-4}\,\mathrm{Re}\big(\overline{g'}h' f^2\big) \tag{6}$$

whenever $f \neq 0$.

Proof. We take $\alpha = p/2$, $u = |f|^2$, and then use (2), (4) and (5) to get the result. $\square$

Proof of Theorem 1. We have to prove that $\Delta(|f|^p) \geqslant 0$, where $p = 4k/(1+k)^2$. Since $p-2<0$, we get from (6) that

$$\begin{aligned}\Delta\bigl(|f|^p\bigr) &\geqslant p^2\bigl(|g'|^2+|h'|^2\bigr)|f|^{p-2}+2p(p-2)|f|^{p-4}|g'|\cdot|h'|\cdot|f|^2\\ &= p^2|g'|^2\bigl(m^2+1\bigr)|f|^{p-2}+2p(p-2)|g'|^2|f|^{p-2}m\\ &= p|g'|^2|f|^{p-2}\bigl[p\bigl(1+m^2\bigr)+2(p-2)m\bigr],\end{aligned}$$

where $m = |h'|/|g'| \leqslant k$. The function $m \mapsto p(1+m^2)+2(p-2)m$ has a negative derivative (because $p<1$ and $m<1$), which implies that

$$\bigl(1+m^2\bigr)p+2(p-2)m \geqslant \bigl(1+k^2\bigr)p+2(p-2)k.$$

On the other hand, $(1+k^2)p+2(p-2)k \geqslant 0$ if and only if $p \geqslant 4k/(1+k)^2$, which proves that $|f|^q$ is subharmonic. To prove that the exponent q is optimal we take $f(z) = z + k\bar{z}$. By (6),

$$\Delta\bigl(|f|^p\bigr)(1) = p^2\bigl(1+k^2\bigr)(1+k)^{p-2}+2p(p-2)(1+k)^{p-2}k.$$

Hence $\Delta(|f|^p)(1) \geqslant 0$ if and only if

$$p\bigl(1+k^2\bigr)+2(p-2)k \geqslant 0,$$

which, as noted above, is equivalent to $p \geqslant q$. This completes the proof of Theorem 1. □

2. Moduli of continuity

For a continuous function $f : \overline{\mathbb{D}} \mapsto \mathbb{C}$ harmonic in $\mathbb{D}$ we define two moduli of continuity

$$\omega(f,\delta) = \sup\bigl\{\bigl|f\bigl(e^{i\theta}\bigr)-f\bigl(e^{it}\bigr)\bigr|:\ \bigl|e^{i\theta}-e^{it}\bigr| \leqslant \delta,\ t,\theta\in\mathbb{R}\bigr\},\quad \delta \geqslant 0,$$

and

$$\tilde{\omega}(f,\delta) = \sup\bigl\{\bigl|f(z)-f(w)\bigr|:\ |z-w| \leqslant \delta,\ z,w\in\overline{\mathbb{D}}\bigr\},\quad \delta \geqslant 0.$$

Clearly $\omega(f,\delta) \leqslant \tilde{\omega}(f,\delta)$, but the reverse inequality need not hold. To see this consider the function

$$f\bigl(re^{i\theta}\bigr) = \sum_{n=1}^{\infty}\frac{(-1)^n r^n\cos n\theta}{n^2},\quad re^{i\theta}\in\overline{\mathbb{D}}.$$

This function is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. The function $v(\theta) = f(e^{i\theta})$, $|\theta|<\pi$, is differentiable and

$$\frac{dv}{d\theta} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin n\theta}{n} = \frac{\theta}{2},\quad |\theta|<\pi.$$

This formula is well known, and can be verified by calculating the Fourier coefficients of the function $\theta \mapsto \theta/2$, $|\theta|<\pi$. It follows that

$$\bigl|f\bigl(e^{i\theta}\bigr)-f\bigl(e^{it}\bigr)\bigr| \leqslant (\pi/2)|\theta-t|,\quad -\pi<\theta,t<\pi,$$

and hence $\omega(f,\delta) \leqslant M\delta$, $\delta>0$, where M is an absolute constant. On the other hand, the inequality $\tilde{\omega}(f,\delta) \leqslant CM\delta$, $C = \text{const.}$, does not hold because it implies that $|\partial f/\partial r| \leqslant CM$, which is not true because

$$\frac{\partial}{\partial r}f\bigl(re^{i\theta}\bigr) = \sum_{n=1}^{\infty}\frac{r^{n-1}}{n},\quad \text{for } \theta=\pi,\ 0<r<1.$$

However, as was proved by Rubel, Shields, and Taylor [2], and Tamrazov [3], if f is a holomorphic function, then $\tilde{\omega}(f,\delta) \leqslant C\omega(f,\delta)$, where C is independent of f and δ. In this note we extend that result to quasiregular harmonic functions.

Theorem 2. *Let f be a k-quasiregular harmonic complex-valued function which has a continuous extension on $\overline{\mathbb{D}}$, then there is a constant C depending only on k such that $\tilde{\omega}(f,\delta) \leqslant C\omega(f,\delta)$.*

In order to deduce this fact from Theorem 1, we need some simple properties of the modulus $\omega(f,\delta)$. Let

$$\omega_0(f,\delta)=\sup\{|f(e^{i\theta})-f(e^{it})|:\ |\theta-t|\leqslant\delta,\ t,\theta\in\mathbb{R}\}.$$

It is easy to check that

$$C^{-1}\omega_0(f,\delta)\leqslant\omega(f,\delta)\leqslant C\omega_0(f,\delta), \tag{7}$$

where C is an absolute constant, and that

$$\omega_0(f,\delta_1+\delta_2)\leqslant\omega_0(f,\delta_1)+\omega_0(f,\delta_2),\quad \delta_1,\delta_2\geqslant 0.$$

Hence $\omega_0(f,2^n\delta)\leqslant 2^n\omega_0(f,\delta)$, and hence $\omega_0(\lambda\delta)\leqslant 2\lambda\omega_0(\delta)$, for $\lambda\geqslant 1$, $\delta\geqslant 0$. From these inequalities and (7) it follows that

$$\omega(f,\lambda\delta)\leqslant 2C\lambda\omega(f,\delta),\quad \lambda\geqslant 1,\ \delta\geqslant 0, \tag{8}$$

and

$$\omega(f,\delta_1+\delta_2)\leqslant C\omega(f,\delta_1)+C\omega(f,\delta_2),\quad \delta_1,\delta_2\geqslant 0, \tag{9}$$

where C is an absolute constant. As a consequence of (8) we have, for $0<p<1$,

$$\int_x^\infty\frac{\omega(f,t)^p}{t^2}\,dt\leqslant C\frac{\omega(f,x)^p}{x},\quad x>0, \tag{10}$$

where C depends only on p. Finally we need the following consequence of the harmonic Schwarz lemma (see [1]).

Lemma 6. *If h is a function harmonic and bounded in the unit disk, with $h(0)=0$, the $|h(\xi)|\leqslant(4/\pi)\|h\|_\infty|\xi|$, for $\xi\in\mathbb{D}$.*

Proof of Theorem 2. It is enough to prove that $|f(z)-f(w)|\leqslant C\omega(f,|z-w|)$ for all $z,w\in\overline{\mathbb{D}}$, where C depends only on k. Assume first that $z=r\in(0,1)$ and $|w|=1$. Then, by Theorem 1, the function $\varphi(\xi)=|f(w)-f(\xi)|^q$, where $q=4k/(1+k)^2<1$, is subharmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, whence

$$\varphi(r)\leqslant\frac{1}{2\pi}\int_{\partial\mathbb{D}}\frac{(1-r^2)\varphi(\zeta)}{|\zeta-r|^2}\,|d\zeta|.$$

Since, by (9),

$$\begin{aligned}\varphi(\zeta)&\leqslant\big(\omega\big(f,|w-r|+|r-\zeta|\big)\big)^q\\&\leqslant C^q\omega\big(f,|w-r|\big)^q+C^q\omega\big(f,|r-\zeta|\big)^q,\end{aligned}$$

we have

$$\begin{aligned}\varphi(z)&\leqslant C^q\omega\big(f,|w-r|\big)^q+\frac{C^q}{2\pi}\int_{\partial\mathbb{D}}\frac{(1-r^2)\omega(f,|r-\zeta|)^q}{|\zeta-r|^2}\,|d\zeta|\\&=C^q\omega\big(f,|w-r|\big)^q+\frac{C^q}{2\pi}\int_{-\pi}^{\pi}\frac{(1-r^2)\omega(|r-e^{it}|)^q}{|e^{it}-r|^2}\,dt.\end{aligned}$$

But simple calculation shows that

$$|r-e^{it}|=\sqrt{(1-r)^2+4r\sin^2(t/2)}\asymp 1-r+|t|\quad\big(0<r<1,\ |t|\leqslant\pi\big).$$

From this, (1), and (10) it follows that

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{(1-r^2)\omega(f,|r-e^{it}|)^q}{|e^{it}-r|^2}\,dt &\leqslant C_1 \int_0^{\pi} \frac{(1-r)\omega(f,1-r+t)^q}{(1-r+t)^2}\,dt \\
&= C_1 \left(\int_0^{1-r} + \int_{1-r}^{\pi} \right) \frac{(1-r)\omega(f,1-r+t)^q}{(1-r+t)^2}\,dt \\
&\leqslant C_2\big(\omega(1-r)\big)^q + C_2(1-r)\int_{1-r}^{\infty} \frac{\omega(f,t)^q}{t^2}\,dt \\
&\leqslant C_3\big(\omega(f,1-r)\big)^q \\
&\leqslant C_4\big(\omega\big(f,|w-z|\big)\big)^q.
\end{aligned}$$

Thus $|f(w)-f(z)| \leqslant C_5\omega(f,|w-z|)$ provided $w \in \partial\mathbb{D}$ and $z \in (0,1)$. By rotation and the continuity of f, we can extend this inequality to the case where $w \in \partial\mathbb{D}$ and $z \in \overline{\mathbb{D}}$.

If $0 < |w| < 1$, we consider the function $h(\xi) = f(\xi w/|w|) - f(\xi z/|w|)$, $|\xi| \leqslant 1$. This function is harmonic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and $h(0) = 0$. Hence, by the harmonic Schwarz lemma, inequality (1), and the preceding case,

$$\begin{aligned}
\big|f(w)-f(z)\big| &= \big|h\big(|w|\big)\big| \\
&\leqslant (4/\pi)|w|\|h\|_\infty \\
&\leqslant C_6|w|\omega\big(f,\big|w/|w|-z/|w|\big|\big) \\
&\leqslant C_7\omega\big(f,|w|\big|w/|w|-z/|w|\big|\big) \\
&= C_7\omega\big(f,|w-z|\big),
\end{aligned}$$

which completes the proof. □

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