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Common fixed point theorems for non-self-mappings in metric spaces of hyperbolic type

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ABSTRACT

In this paper, the concept of a pair of non-linear contraction type mappings in a metric space of hyperbolic type is introduced and the conditions guaranteeing the existence of a common fixed point for such non-linear contractions are established. Presented results generalize and improve some of the known results. An example is constructed to show that our theorems are genuine generalizations of the main theorems of Assad, Ćirić, Khan et al., Rhoades and Imdad and Kumar. One of the possible applications of our results is also presented.

1. Introduction

Fixed point theorems for contraction self-mappings have find applications in diverse disciplines of mathematics, engineering and economics. In convex spaces occur cases where the involved function is not necessarily a self-mapping of a closed subset. Assad [1] and Assad and Kirk [2] first studied non-self-contraction mappings in a metric space (X, d), metrically convex in the sense of Menger (that is, for each x, y in X with $x \neq y$ there exists z in X, $x \neq z \neq y$, such that $d(x, z) + d(z, y) = d(x, y)$). In recent years, this technique has been developed and fixed and common fixed points of non-self-mappings have been studied by many authors [3–14]. Some of the obtained results have find applications (c.f. [2,14–16]). In numerical mathematics, a restricted condition $T(\partial K) \subseteq K$ is especially favorable instead of $T(K) \subseteq K$, where K is a closed subset of X, $T : K \to X$ and ∂K is the boundary of K.

In an attempt to generalize a theorem of Assad [1] and Assad and Kirk [2], Rhoades [13] proved the following result in a Banach space.

Theorem 1. *Let X be a Banach space, K a non-empty closed subset of X and $T : K \to X$ a mapping of K into X satisfying the condition*

$$d(Tx, Ty) \leq h \max\left\{\frac{d(x, y)}{2}, d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q}\right\} \tag{1}$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$ and T has the additional property that for each $x \in \partial K$, the boundary of K, $Tx \in K$, then T has a unique fixed point.

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Recently Imdad and Kumar [10] generalized the result of Rhoades [13] and Ćirić [3]. They proved the following theorem.

Theorem 2. *Let X be a Banach space, K a non-empty closed subset of X and $F, T : K \to X$ two mappings satisfying the condition*

$$d(Fx, Fy) \le h \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \frac{d(Tx, Fy) + d(Ty, Fx)}{q}\right\} \tag{2}$$

for all x, y in K, $0 < h < 1$, $q \ge 1 + 2h$ and

(i) $\partial K \subseteq TK$, $FK \cap K \subset TK$,

(ii) $Tx \in \partial K \Longrightarrow Fx \in K$, *and*

(iii) TK *is closed in* X.

Then there exists a coincidence point z in X. Moreover, if F and T are coincidentally commuting, then z remains a unique common fixed point of F and T.

Recall (see [17]) that a pair of mappings (F, T), defined on a non-empty set S, is said to be coincidentally commuting, if $Tx = Fx \Longrightarrow FTx = TFx$; $x \in S$.

The purpose of this paper is to introduce the concept of a new pair of non-linear contractive type non-self-mappings which satisfy a new contractive condition, weaker than (2) and to prove common fixed point theorems in metric spaces of hyperbolic type. Our theorems generalize the main theorems of Assad [1], Ćirić [3], Khan et al. [12], Rhoades [13] and Imdad and Kumar [10] in many aspects. An example is constructed to show that our results are genuine generalizations of the known results from this area. One of the possible applications of our results is also presented.

2. Main results

Throughout our consideration we suppose that (X, d) is a convex metric space which contains a family L of metric segments (isometric images of real line segments) such that

(a) each two points x, y in X are end points of exactly one member $seg[x, y]$ of L, where

$$seg[x, y] = \{z \in X : d(z, x) = \lambda d(x, y) \text{and } d(z, y) = (1 - \lambda)d(x, y);\ \lambda \in [0, 1]\},$$

and

(b) if u, x, y in X and if $z \in seg[x, y]$ satisfies $d(x, z) = \lambda d(x, y)$ for any $\lambda \in [0, 1]$, then

$$d(u, z) \le (1 - \lambda)d(u, x) + \lambda d(u, y). \tag{3}$$

A space of this type is said to be *a metric space of hyperbolic type* (Takahashi [18] uses the term *a convex metric space*). This class includes all normed linear spaces, as well as all spaces with hyperbolic metric (see [19] for a discussion). For instance, if X is a Banach space, then

$$seg[x, y] = \{(1 - \lambda)x + \lambda y : 0 \le \lambda \le 1\}.$$

A linear space with a translation invariant metric satisfying

$$d(\lambda x + (1 - \lambda)y, 0) \le \lambda d(x, 0) + (1 - \lambda)d(y, 0)$$

is also a metric space of hyperbolic type. There are many other examples but we consider these as paradigmatic.

Now we shall prove a common fixed point theorem for a new pair of non-linear contraction type mappings in a metric space of hyperbolic type. Recall that the concept of a non-linear contraction was introduced and studied in [20], and that some applications of non-linear contractions was considered in [21].

Theorem 3. *Let X be a metric space of hyperbolic type, K a non-empty closed subset of X and ∂K the boundary of K. Let ∂K be non-empty and let $T : K \to X$ and $F : K \cap T(K) \to X$ be two non-self-mappings satisfying the following conditions:*

$$d(Fx, Fy) \le \varphi\left(\max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \min\{d(Tx, Fy), d(Ty, Fx)\}, \frac{d(Tx, Fy) + d(Ty, Fx)}{3}\right\}\right), \tag{4}$$

for all x, y in X, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a real function which has the following properties:

$\varphi(t+) < t$ for $t > 0$ and $\varphi(t)$ is non-decreasing.

Suppose that F and T have the additional properties:

(i) $\partial K \subseteq TK$;

(ii) $FK \cap K \subseteq TK$;

(iii) $Tx \in \partial K \Rightarrow Fx \in K$;

(iv) $K \cap T(K)$ *is complete.*

Then there exists a coincidence point z in X. Moreover, if F and T are coincidentally commuting, then z is a unique common fixed point of F and T.

Proof. If $F(K), T(K) \subseteq K$, then theorem holds without the hypotheses of convexity of X and under a contractive condition weaker then the condition (4). The proof in that case is much simpler, since Cases 2 and 3, do not occur. We will give a proof under hypothesis that each of the mapping F and T is not necessarily a self-mapping.

Let $x \in \partial K$ be arbitrary. We construct three sequences: $\{x_n\}$ and $\{z_n\}$in K and a sequence $\{y_n\}$in $FK \subseteq X$ in the following way. Set $z_0 = x$. Since $z_0 \in \partial K$, by (i) there exists a point $x_0 \in K$ such that $Tx_0 = z_0$. Then, since $Tx_0 \in \partial K$, from (iii) we conclude that $Fx_0 \in K$. So from (ii), $Fx_0 \in TK$. Thus, there exists $x_1 \in K$ such that $Tx_1 = Fx_0 \in K$. Set $z_1 = y_1 = Fx_0 = Tx_1$ and $y_2 = Fx_1$.

If $y_2 \in FK \cap K$, then from (ii), $y_2 \in TK$ and so there is a point $x_2 \in K$ such that $Tx_2 = y_2 = z_2 = Fx_1$.

If $y_2 = Fx_1 \notin K$, then by z_2 we denote a point in ∂K $(z_2 \neq y_2)$ such that $z_2 \in seg\,[y_1, y_2] = seg\,[Fx_0, Fx_1]$. By (i), there is $x_2 \in K$ such that $Tx_2 = z_2$. Thus

$$z_2 \in \partial K \cap seg\,[Fx_0, Fx_1]\,.$$

Now we set $y_3 = Fx_2 = z_3$. Since $Fx_2 \in FK \cap K \subseteq TK$, from (ii) there is a point $x_3 \in K$ such that $Tx_3 = y_3$.

Note that in the case: $z_2 \neq y_2 = Fx_1$, we have $z_1 = y_1 = Fx_0$ and $z_3 = y_3 = Fx_2$.

Continuing the foregoing procedure we construct three sequences:
$\{x_n\} \subseteq K$, $\{z_n\} \subseteq K$ and $\{y_n\} \subseteq FK \subseteq X$ such that:
(a) $y_n = Fx_{n-1}$;
(b) $z_n = Tx_n$;
(c) $z_n = y_n$ if and only if $y_n \in K$;
(d) $z_n \neq y_n$ whenever $y_n \notin K$ and then $z_n \in \partial K$ is such that

$$z_n \in \partial K \cap seg\,[Fx_{n-2}, Fx_{n-1}]\,.$$

Observation. If $z_n \neq y_n$, then $z_n \in \partial K$, which then implies, by (b), (iii) and (a), that $z_{n+1} = y_{n+1} \in K$. Also, $z_n \neq y_n$ implies that $z_{n-1} = y_{n-1} \in K$, since otherwise $z_{n-1} \in \partial K$, which then implies $z_n = y_n \in K$.

Now we wish to estimate $d(z_n, z_{n+1})$. If $d(z_n, z_{n+1}) = 0$ for some n, then it is easy to show that $d(z_n, z_{n+k}) = 0$ for all $k \geq 1$.

Suppose that $d(z_n, z_{n+1}) > 0$ for all n. From the above observation we conclude that there are three possibilities: (1) $z_n = y_n \in K$ and $z_{n+1} = y_{n+1}$; (2) $z_n = y_n \in K$, but $z_{n+1} \neq y_{n+1}$, and (3) $z_n \neq y_n$, in which case $z_n \in \partial K \cap seg\,[Fx_{n-2}, Fx_{n-1}]$.

Case 1. Let $z_n = y_n \in K$ and $z_{n+1} = y_{n+1} \in K$. Then $z_n = y_n = Fx_{n-1}$, $z_{n+1} = y_{n+1} = Fx_n$ and $z_{n-1} = Tx_{n-1}$ (observe that not necessarily $z_{n-1} = y_{n-1}$). Then from (4),

$$\begin{aligned} d(z_n, z_{n+1}) &= d(Fx_{n-1}, Fx_n) \\ &\leq \varphi\Bigg(\max\Bigg\{\frac{d(Tx_{n-1}, Tx_n)}{2}, d(Tx_{n-1}, Fx_{n-1}), d(Tx_n, Fx_n), \min\{d(Tx_{n-1}, Fx_n), d(Tx_n, Fx_{n-1})\}, \\ &\qquad \frac{d(Tx_{n-1}, Fx_n) + d(Tx_n, Fx_{n-1})}{3}\Bigg\}\Bigg) \\ &= \varphi\left(\max\left\{\frac{d(z_{n-1}, z_n)}{2}, d(z_{n-1}, z_n), d(z_n, z_{n+1}), 0, \frac{d(z_{n-1}, z_{n+1})}{3}\right\}\right) \\ &\leq \varphi(\max\{d(z_{n-1}, z_n), d(z_n, z_{n+1})\}). \end{aligned} \tag{5}$$

Hence, as $\varphi(t) < t$ for $t > 0$,

$$d(z_n, z_{n+1}) \leq \varphi(d(z_{n-1}, z_n)). \tag{6}$$

From (6) immediately follows, as $z_n = y_n$, $z_{n+1} = y_{n+1}$,

$$d(y_n, y_{n+1}) \leq \varphi(d(z_{n-1}, z_n)). \tag{7}$$

Note that (7) holds whenever $y_n = z_n$, without regard to $y_{n+1} = z_{n+1}$, or $y_{n+1} \neq z_{n+1}$.

Case 2. Let $z_n = y_n \in K$, but $z_{n+1} \neq y_{n+1}$. Then $z_{n+1} \in \partial K \cap seg\,[y_n, y_{n+1}]$.

Note that from (3) with $u = y$, we get

$$d(y, z) \leq (1 - \lambda)d(x, y).$$

Thus we have

$$d(x, y) \leq d(x, z) + d(z, y) \leq \lambda d(x, y) + (1 - \lambda)d(x, y) = d(x, y).$$

Therefore

$$z \in seg\,[x, y] \Longrightarrow d(x, z) + d(z, y) = d(x, y). \tag{8}$$

From (8), because $z_{n+1} \in seg\,[y_n, y_{n+1}] = seg\,[z_n, y_{n+1}]$,

$$d(z_n, z_{n+1}) = d(y_n, z_{n+1}) = d(y_n, y_{n+1}) - d(z_{n+1}, y_{n+1}) < d(y_n, y_{n+1}).$$

Thus, by (7),

$$d(z_n, z_{n+1}) \le \varphi(d(z_{n-1}, z_n)). \tag{9}$$

Case 3. Let $z_n \ne y_n$. Then

$$z_n \in \partial K \cap \operatorname{seg}[y_{n-1}, y_n] \tag{10}$$

and, by Observation, $z_{n+1} = y_{n+1}$ and $z_{n-1} = y_{n-1}$.

Note that from (3) it follows that, if $z \in seg[x, y]$, then for any u in X,

$$d(u, z) \le \max\{d(u, x), d(u, y)\}. \tag{11}$$

From (11) with $x = y_{n-1} = Fx_{n-2}$, $y = y_n = Fx_{n-1}$, $z = z_n$ and $u = z_{n+1} = Fx_n$, and by (a), we get

$$d(z_n, z_{n+1}) \le \max\{d(Fx_n, Fx_{n-2}), d(Fx_n, Fx_{n-1})\}. \tag{12}$$

Consider at first the case

$$d(z_n, z_{n+1}) \le d(Fx_{n-1}, Fx_n).$$

Since $z_{n+1} = y_{n+1}, z_{n-1} = y_{n-1}$ and $d(z_{n-1}, y_n) = d(z_{n-1}, z_n) + d(z_n, y_n)$, from (4) we get

$$\begin{aligned} d(z_n, z_{n+1}) &\le d(Fx_{n-1}, Fx_n) \\ &\le \varphi\left(\max\left\{\frac{d(Tx_{n-1}, Tx_n)}{2}, d(Tx_{n-1}, Fx_{n-1}), d(Tx_n, Fx_n), \min\{d(Tx_{n-1}, Fx_n), d(Tx_n, Fx_{n-1})\},\right.\right. \\ &\qquad \left.\left.\frac{d(Tx_{n-1}, Fx_n) + d(Tx_n, Fx_{n-1})}{3}\right\}\right) \\ &= \varphi\left(\max\left\{\frac{d(z_{n-1}, z_n)}{2}, d(z_{n-1}, y_n), d(z_n, z_{n+1}), d(z_n, y_n), \frac{d(z_{n-1}, z_{n+1}) + d(z_n, y_n)}{3}\right\}\right) \\ &\le \varphi\left(\max\left\{d(y_{n-1}, y_n), d(z_n, z_{n+1}), \frac{2}{3}\max\{d(y_{n-1}, y_n), d(z_n, z_{n+1})\}\right\}\right). \end{aligned}$$

Hence, as $\varphi(t) < t$ for $t > 0$,

$$d(z_n, z_{n+1}) \le \varphi(d(y_{n-1}, y_n)).$$

Since $z_{n-1} = y_{n-1}$, then from (7) (with $y_n = y_{n-1}, y_{n+1} = y_n$) we have $d(y_{n-1}, y_n) \le \varphi(d(z_{n-2}, z_{n-1}))$. Therefore, we conclude that in this case

$$d(z_n, z_{n+1}) \le \varphi(d(z_{n-2}, z_{n-1})). \tag{13}$$

Consider now the second possibility, that is the case

$$d(z_n, z_{n+1}) \le d(Fx_{n-2}, Fx_n). \tag{14}$$

Then from (4),

$$\begin{aligned} d(Fx_{n-2}, Fx_n) &\le \varphi\left(\max\left\{\frac{d(Tx_{n-2}, Tx_n)}{2}, d(Tx_{n-2}, Fx_{n-2}), d(Tx_n, Fx_n),\right.\right. \\ &\qquad \left.\left.\min\{d(Tx_{n-2}, Fx_n), d(Tx_n, Fx_{n-2})\}, \frac{d(Tx_{n-2}, Fx_n) + d(Tx_n, Fx_{n-2})}{3}\right\}\right) \\ &= \varphi\left(\max\left\{\frac{d(z_{n-2}, z_n)}{2}, d(z_{n-2}, z_{n-1}), d(z_n, z_{n+1}), d(z_n, z_{n-1}), \frac{d(z_{n-2}, z_{n+1}) + d(z_n, z_{n-1})}{3}\right\}\right). \end{aligned} \tag{15}$$

Since $y_{n-1} = z_{n-1}, z_{n+1} = y_{n+1}$ and $z_n \in \partial K \cap seg\,[y_{n-1}, y_n]$, from (8) and (7) we obtain

$$\begin{aligned} &d(z_n, z_{n-1}) = d(z_n, y_{n-1}) < d(y_{n-1}, y_n) \le \varphi(d(z_{n-2}, z_{n-1})) < d(z_{n-2}, z_{n-1}); \\ &d(z_{n-2}, z_n) \le d(z_{n-2}, z_{n-1}) + d(y_{n-1}, z_n) < 2d(z_{n-2}, z_{n-1}); \\ &d(z_{n-2}, z_{n+1}) \le d(z_{n-2}, y_{n-1}) + d(y_{n-1}, y_{n+1}) \\ &\qquad = d(z_{n-2}, z_{n-1}) + d(Fx_{n-2}, Fx_n). \end{aligned}$$

Now, from (15) and (14),

$$d(Fx_{n-2}, Fx_n) \le \varphi\left(\max\left\{d(z_{n-2}, z_{n-1}), d(Fx_{n-2}, Fx_n), \frac{d(z_{n-2}, z_{n-1}) + d(Fx_{n-2}, Fx_n) + d(z_{n-2}, z_{n-1})}{3}\right\}\right). \tag{16}$$

Because $\varphi(t) < t$ for $t > 0$, then $d(Fx_{n-2}, Fx_n) \leq \varphi(d(Fx_{n-2}, Fx_n))$ is impossible. If from (16)

$$d(Fx_{n-2}, Fx_n) \leq \varphi\left(\frac{2d(z_{n-2}, z_{n-1}) + d(Fx_{n-2}, Fx_n)}{3}\right),$$

then

$$d(Fx_{n-2}, Fx_n) < \frac{2d(z_{n-2}, z_{n-1}) + d(Fx_{n-2}, Fx_n)}{3}.$$

Hence $d(Fx_{n-2}, Fx_n) < d(z_{n-2}, z_{n-1})$ and then it implies that $[2d(z_{n-2}, z_{n-1}) + d(Fx_{n-2}, Fx_n)]/3 < d(z_{n-2}, z_{n-1})$. Thus from (16), $d(Fx_{n-2}, Fx_n) \leq \varphi(d(z_{n-2}, z_{n-1}))$. So by (14) we obtain

$$d(z_n, z_{n+1}) \leq \varphi(d(z_{n-2}, z_{n-1})). \tag{17}$$

From (6), (9), (13) and (17) we conclude that in all cases

$$d(z_n, z_{n+1}) \leq \varphi(\max\{d(z_{n-2}, z_{n-1}), d(z_{n-1}, z_n)\}). \tag{18}$$

Now we shall show, by induction, that for all $n \geq 0$,

$$d(z_n, z_{n+1}) \leq \varphi^{[\frac{n}{2}]}(\delta), \tag{19}$$

where $[k]$ denotes the greatest integer which does not exceed k and

$$\delta = \max\{d(z_0, z_1), d(z_1, z_2)\}.$$

For $n = 0$ we have,

$$d(z_0, z_1) \leq \max\{d(z_0, z_1), d(z_1, z_2)\} = \delta = \varphi^0(\delta) = \varphi^{[\frac{0}{2}]}(\delta).$$

Similarly, for $n = 1$,

$$d(z_1, z_2) \leq \max\{d(z_0, z_1), d(z_1, z_2)\} = \delta = \varphi^0(\delta) = \varphi^{[\frac{1}{2}]}(\delta).$$

Therefore, (19) holds for $n = 0$ and $n = 1$. Suppose now that (19) holds for some n and $n + 1$. Then from (18),

$$\begin{aligned} d(z_{n+2}, z_{n+3}) &\leq \varphi(\max\{d(z_n, z_{n+1}), d(z_{n+1}, z_{n+2})\}) \\ &\leq \varphi\left(\max\left\{\varphi^{[\frac{n}{2}]}(\delta), \varphi^{[\frac{n+1}{2}]}(\delta)\right\}\right) \\ &= \max\left\{\varphi^{[\frac{n}{2}]+1}(\delta), \varphi^{[\frac{n+1}{2}]+1}(\delta)\right\} \\ &\leq \varphi^{[\frac{n+2}{2}]}(\delta). \end{aligned}$$

Thus by induction we conclude that (19) holds for all $n \geq 0$. Taking the limit in (19) as $n \to \infty$ we get

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0. \tag{20}$$

Now we shall show that $\{z_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be arbitrary. Because $\varphi(t+) < t$ for $t > 0$, then $\epsilon - \varphi(\epsilon+) > 0$ and there is a $\delta > 0$ (we may suppose that $\delta < \epsilon/2$) such that

$$\varphi(t) < \frac{\epsilon + \varphi(\epsilon+)}{2} \text{for each } t < \epsilon + \delta. \tag{21}$$

From (20) there is a positive integer n_0 such that for all $n - 1 \geq n_0$,

$$d(z_n, z_{n+1}) \leq \min\left\{\frac{\epsilon - \varphi(\epsilon+)}{4}, \frac{\delta}{2}\right\}. \tag{22}$$

By induction, we shall prove that for any $m > n \geq n_0$:

$$d(z_n, z_m) < \epsilon. \tag{23}$$

Let $n \geq n_0 + 1$ be arbitrary. Then from (22) the inequality (23) holds for $m = n + 1$. Suppose now that (23) holds for some $m > n$. Observe that for any k there are two possibilities: $z_k = y_k$, or $z_k \neq y_k$, but if $z_k \neq y_k$, then $z_{k+1} = y_{k+1}$.

Consider the case $z_n = y_n, z_m = y_m$, and $z_{m+1} \neq y_{m+1}$. Then $z_n = Fx_{n-1}, z_{m+2} = y_{m+2} = Fx_{m+1}$. So from (4),

$$\begin{aligned} d(z_n, z_{m+2}) &= d(Fx_{n-1}, Fx_{m+1}) \\ &\le \varphi\left(\max\left\{\frac{d(Tx_{n-1}, Tx_{m+1})}{2}, d(Tx_{n-1}, Fx_{n-1}), d(Tx_{m+1}, Fx_{m+1}),\right.\right. \\ &\qquad \left.\left.\min\{d(Tx_{n-1}, Fx_{m+1}), d(Tx_{m+1}, Fx_{n-1})\}, \frac{d(Tx_{n-1}, Fx_{m+1}) + d(Tx_{m+1}, Fx_{n-1})}{3}\right\}\right) \\ &= \varphi\left(\max\left\{\frac{d(z_{n-1}, z_{m+1})}{2}, d(z_{n-1}, z_n), d(z_{m+1}, z_{m+2}), d(z_{m+1}, z_n), \frac{d(z_{n-1}, z_{m+2}) + d(z_{m+1}, z_n)}{3}\right\}\right). \end{aligned} \tag{24}$$

By (22) and (23),

$$\begin{aligned} &d(z_{n-1}, z_{m+1}) \le d(z_{n-1}, z_n) + d(z_n, z_m) + d(z_m, z_{m+1}) < \frac{\delta}{2} + \epsilon + \frac{\delta}{2} = \epsilon + \delta; \\ &d(z_{m+1}, z_n) \le d(z_n, z_m) + d(z_m, z_{m+1}) \le \epsilon + \frac{\delta}{2}; \\ &d(z_{n-1}, z_{m+2}) + d(z_{m+1}, z_n) \le d(z_{n-1}, z_{m+1}) + d(z_{m+1}, z_{m+2}) + d(z_n, z_m) + d(z_m, z_{m+1}) \\ &\qquad \le \epsilon + \delta + \frac{\delta}{2} + \epsilon + \frac{\delta}{2} < 3\epsilon. \end{aligned}$$

Thus

$$\begin{aligned} t_M &= \max\left\{\frac{d(z_{n-1}, z_{m+1})}{2}, d(z_{n-1}, z_n), d(z_{m+1}, z_{m+2}), d(z_{m+1}, z_n), \frac{d(z_{n-1}, z_{m+2}) + d(z_{m+1}, z_n)}{3}\right\} \\ &< \epsilon + \delta. \end{aligned}$$

Then from (24), $d(z_n, z_{m+2}) \le \varphi(t_M)$, and by (21),

$$d(z_n, z_{m+2}) < \frac{\epsilon + \varphi(\epsilon+)}{2}. \tag{25}$$

Now, by (25) and (22),

$$\begin{aligned} d(z_n, z_{m+1}) &\le d(z_n, z_{m+2}) + d(z_{m+2}, z_{m+1}) \\ &< \frac{\epsilon + \varphi(\epsilon+)}{2} + \frac{\epsilon - \varphi(\epsilon+)}{4} < \epsilon. \end{aligned}$$

Therefore we proved (23) for the case $z_n = y_n, z_m = y_m$, and $z_{m+1} \neq y_{m+1}$. The rest of the cases can be proved on the similar lines of proofs. Thus, we proved (23).

From (23) we conclude that $\{z_n\}$ is the Cauchy sequence. Since $z_n = Tx_n \in K \cap TK$ and $K \cap TK$ is complete, there is some point $z \in K \cap TK$ such that

$$\lim_{n\to\infty} z_n = z.$$

Let w in K be such that $Tw = z$. By construction of $\{z_n\}$, there is a subsequence $\{z_{n(k)}\}$ such that $z_{n(k)} = y_{n(k)} = Fx_{n(k)-1}$ and so $\lim_{k\to\infty} Fx_{n(k)-1} = z$.

From (4),

$$\begin{aligned} d(Fx_{n(k)-1}, Fw) &\le \varphi\left(\max\left\{\frac{d(Tx_{n(k)-1}, Tw)}{2}, d(Tx_{n(k)-1}, Fx_{n(k)-1}), d(Tw, Fw), \min\{d(Tx_{n(k)-1}, Fw),\right.\right. \\ &\qquad \left.\left. d(Tw, Fx_{n(k)-1})\}, \frac{d(Tx_{n(k)-1}, Fw) + d(Tw, Fx_{n(k)-1})}{3}\right\}\right) \\ &= \varphi\left(\max\left\{\frac{d(z_{n(k)-1}, z)}{2}, d(z_{n(k)-1}, z_{n(k)}), d(z, Fw), d(z, z_{n(k)}), \frac{d(z_{n(k)-1}, Fw) + d(z, z_{n(k)})}{3}\right\}\right). \end{aligned}$$

Taking the limit as $k \to \infty$ we get $d(z, Fw) \le \varphi(d(z, Fw))$. If we suppose that $d(z, Fw) > 0$, then we have

$$d(z, Fw) \le \varphi(d(z, Fw)) < d(z, Fw),$$

a contradiction, as $\varphi(t) < t$ for $t > 0$. Thus $d(z, Fw) = 0$ and hence $Fw = z$. Therefore, $Fw = Tw$ which shows that w is a point of coincidence for F and T.

Suppose now that F and T are coincidentally commuting. Then

$$z = Fw = Tw \Longrightarrow Fz = FTw = TFw = Tz.$$

Therefore z is a point of coincidence for F and T. To prove that z is a common fixed point of F and T, suppose to the contrary, that $d(Fz, z) > 0$. Then from (4),

$$\begin{aligned} d(Fz, z) &= d(Fz, Fw) \\ &\leq \varphi\left(\max\left\{\frac{d(Fz, z)}{2}, 0, 0, d(Fz, z), \frac{d(Fz, z) + d(z, Fz)}{3}\right\}\right), \end{aligned}$$

and hence $d(Fz, z) = 0$; hence $Fz = z$. Thus we proved that

$$Fz = z = Tz. \quad \blacksquare$$

For the introduced pair of mappings we have the following result in a Banach space.

Theorem 4. *Let X be a Banach space, K a non-empty closed subset of X and ∂K the boundary of K. Let ∂K be non-empty and let $T : K \to X$ and $F : K \cap T(K) \to X$ be two non-self-mappings satisfying the following conditions:*

$$d(Fx, Fy) \leq h \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \min\{d(Tx, Fy), d(Ty, Fx)\}, \frac{d(Tx, Fy) + d(Ty, Fx)}{q}\right\}, \tag{26}$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$ and

(i) $\partial K \subseteq TK$;
(ii) $FK \cap K \subseteq TK$;
(iii) $Tx \in \partial K \Rightarrow Fx \subseteq K$;
(iv) $K \cap T(K)$ *is complete.*

Then there exists a coincidence point z in X. Moreover, if F and T are coincidentally commuting, then z is a unique common fixed point of F and T.

Proof. It is easy to see that (26) implies that

$$d(Fx, Fy) \leq h_1 \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \min\{d(Tx, Fy), d(Ty, Fx)\}, \frac{d(Tx, Fy) + d(Ty, Fx)}{3}\right\}, \tag{27}$$

where $h_1 = 3h/(1 + 2h)$. Taking in (4) $\varphi(t) = h_1 t$; $0 < h_1 < 1$, then we obtain (27). Therefore Theorem 4 is a special case of Theorem 3. $\blacksquare$

Remark 1. Theorems 3 and 4 generalize Theorem 2 of Imdad and Kumar [10], and therefore Theorem 1 of Rhoades [13] and Theorem of Ćirić [3].

Remark 2. The following simple example shows that Theorem 3 is essentially more general than Theorem 1 in [13] and Theorem 2 in [10].

Example. Let X be the set of real numbers with the usual metric, $K = [0, +\infty)$ and let $T : K \to X$ and $F : K \cap T(K) \to X$ be two non-self-mappings defined by

$$\begin{aligned} Fx &= \frac{x^2}{2 + x^2}; \\ Tx &= x^2. \end{aligned}$$

Define a governing function $\varphi(t)$ by

$$\varphi(t) = \frac{t}{1 + t} \quad \text{for } t \geq 0.$$

Then for any $x, y \in K$,

$$d(Fx, Fy) = \frac{2|x^2 - y^2|}{4 + 2(x^2 + y^2) + x^2y^2} \leq \frac{|x^2 - y^2|}{2 + |x^2 - y^2|} = \frac{|x^2 - y^2|/2}{1 + |x^2 - y^2|/2} = \varphi\left(\frac{d(Tx, Ty)}{2}\right).$$

Therefore F and T satisfy (4). Since $\varphi(t)$ is continuous, non-decreasing and $\varphi(t) < t$ for $t > 0$, all hypotheses in Theorem 3 are satisfied and F and T have a unique common fixed point $z = 0$.

To see that Theorem 2 of Imdad and Kumar [10] is not applicable, let $h \in (0, 1)$ be any fixed number. Then for $y = 0$ and any $0 < x < \sqrt{(1 - h)/h}$ we have

$$\begin{aligned} d(Fx, F0) &= \frac{x^2}{2 + x^2} > h\frac{1 + x^2}{2 + x^2}x^2 = h \max\left\{\frac{x^2}{2}, \frac{1 + x^2}{2 + x^2}x^2, 0, \frac{x^2 + \frac{x^2}{2+x^2}}{q}\right\} \\ &= h \max\left\{\frac{d(Tx, T0)}{2}, d(Tx, Fx), d(T0, F0), \frac{d(Tx, F0) + d(T0, Fx)}{q}\right\}. \end{aligned}$$

Therefore, the inequality (2) in Theorem 2 is not satisfied.

3. Applications

To demonstrate an application of our results, we prove the following theorem.

Theorem 5. *Let K be a non-empty subset of a normed space X and F and T be mappings of K into X satisfying conditions* (i)–(iii) *of* Theorem 4 *and*

$$d(Fx, Fy) \leq \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \min\{d(Tx, Fy), d(Ty, Fx)\}, \frac{d(Tx, Fy) + d(Ty, Fx)}{3}\right\}, \tag{28}$$

for all x, y in K. Suppose that K is q-starshaped for some $q \in K$. Then F and T have a unique common fixed point in K, provided one of the following conditions holds:

(i) *$K \cap T(K)$ is compact and F and T are continuous,*
(ii) *X is complete, $K \cap T(K)$ is weakly compact, T is weakly continuous and $I - F$ is demiclosed at* 0*,*
(iii) *X is complete, $K \cap T(K)$ is weakly compact, T is weakly continuous and F is completely continuous.*

Proof. Let $q \in K$ be such that

$$(1-\lambda)q + \lambda x \in K \quad \text{for all } x \in K \text{and all } \lambda \in (0, 1).$$

Define $F_n : K \to X$ by

$$F_n x = (1 - k_n)q + k_n Fx \tag{29}$$

for all $x \in K$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. It is easy to show that for each $n \geq 1$, F_n and T satisfy conditions (i)–(iii) of Theorem 4, (27) with $h = k_n$.

(i) Since $K \cap T(K)$ is compact, we can apply Theorem 4 for F_n and T. Thus, for each $n \geq 1$, there exists $x_n \in K$ such that $x_n = Tx_n = F_n x_n$. The compactness of $K \cap T(K)$ implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to z \in K$ as $m \to \infty$. Then by the continuity of T, $x_m = Tx_m \to Tz$. Also by the continuity of F, $Fx_m \to Fz$. Since $k_m \to 1$, $x_m = F_m x_m = (1 - k_m)q + k_m Fx_m \to Fz$. Therefore, we proved that

$$Fz = z = Tz.$$

(ii) Since $K \cap T(K)$ is weakly compact and weak topology is Hausdorff, we conclude that $K \cap T(K)$ is weakly closed and so strongly closed. Therefore, $K \cap T(K)$ is complete, as X is complete. By Theorem 4, for each $n \geq 1$ there exists $x_n \in K$ such that $x_n = Tx_n = F_n x_n$. Since $\{x_n\} \subseteq K \cap T(K)$ and $K \cap T(K)$ is weakly compact, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ and $y \in K$ such that $x_m \to y$ weakly. From (29) we have

$$\begin{aligned}(I - F)x_m &= x_m - Fx_m \\ &= x_m - k_m^{-1} F_m x_m - (1 - k_m)q \\ &= (1 - k_m^{-1})x_m + (k_m^{-1} - 1)q.\end{aligned}$$

Hence

$$\|(I - F)x_m\| \leq (k_m^{-1} - 1)(\|x_m\| + \|q\|). \tag{30}$$

Since weakly convergent sequences are norm bounded and as $k_m^{-1} \to 1$ as $m \to \infty$, from (30) we conclude that $(I - F)x_m \to 0 \in K$. Further, as $x_m \to y \in K$ and $(I - F)$ is demiclosed at 0, it follows that $(I - F)y = 0$. Hence $Fy = y$. Since $Tx_m = x_m \to y$, weak continuity of T implies that $y = Ty$. Therefore we proved that

$$Fy = y = Ty.$$

(iii) As in (ii), we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in K converging weakly to $y \in K$ as $m \to \infty$. Since F is completely continuous, $Fx_m \to Fy$ as $m \to \infty$. Since $k_m \to 1$ and $Fx_m \to Fy$ as $m \to \infty$, we have $x_m = F_m x_m = k_m Fx_m + (1-k_m)q \to Fy$ as $m \to \infty$. Thus $y = Fy$. Since $Tx_m = x_m \to y$, weak continuity of T implies that $y = Ty$. Therefore, we proved that

$$Fz = z = Tz. \quad \blacksquare$$

Note that as another application, certain invariant approximation results for introduced class of mappings can be also derived.

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References

[1] N.A. Assad, On a fixed point theorem in Banach spaces, Tamkang J. Math. 7 (1976) 91–94.
[2] N.A. Assad, W.A. Kirk, Fixed-point theorems for set-valued mappings of contractive type, Pacific J. Math. 43 (1972) 553–562.

[3] Lj.B. Ćirić, A remark on Rhoades fixed point theorem for non-self mappings, Int. J. Math. Math. Sci. 16 (1993) 397–400.
[4] Lj.B. Ćirić, Quasi-contraction non-self mappings on Banach spaces, Bull. Acad. Serbe Sci. Arts 23 (1998) 25–31.
[5] Lj.B. Ćirić, J.S. Ume, M.S. Khan, H.K. Pathak, On some non-self mappings, Math. Nachr. 251 (2003) 28–33.
[6] L.B. Ćirić, J.S. Ume, Multi-valued non-self mappings on convex metric spaces, Nonlinear Anal. 60 (2005) 1053–1063.
[7] L.B. Ćirić, Contractive-type non-self mappings on metric spaces of hyperbolic type, J. Math. Anal. Appl. 317 (2006) 28–42.
[8] Lj.B. Ćirić, J.S. Ume, Some common fixed point theorems for weakly compatible mappings, J. Math. Anal. Appl. 314 (2) (2006) 488–499.
[9] Lj. Gajić, V. Rakočević, Pair of non-self-mappings and common fixed points, Appl. Math. Comput. 187 (2) (2007) 999–1006.
[10] M. Imdad, S. Kumar, Rhoades-type fixed-point theorems for a pair of nonself mappings, Comput. Math. Appl. 46 (2003) 919–927.
[11] M.S. Khan, Common fixed point theorems for multivalued mappings, Pacific J.Math. 95 (1981) 337–347.
[12] M.S. Khan, H.K. Pathak, M.D. Khan, Some fixed point theorems in metrically convex spaces, Georgian Math. J. 7 (2000) 523–530.
[13] B.E. Rhoades, A fixed point theorem for some non-self mappings, Math. Japonica 23 (1978) 457–459.
[14] T. Tsachev, V.G. Angelov, Fixed points of non-self mappings and applications, Nonlinear Anal. 21 (1993) 9–16.
[15] B.E. Rhoades, Some applications of contractive type mappings, Math. Sem. Notes Kobe Univ. 5 (2) (1977) 137–139.
[16] L.C. Zeng, T. Tanaka, J.C. Yao, Iterative construction of fixed points of non-self mappings in Banach spaces, J. Comput. Appl. Math. 206 (2) (2007) 814–825.
[17] G. Jungck, B.E. Rhoades, Fixed points for set-valued functions without continuity, Indian J. Pure Appl. Math. 29 (3) (1998) 227–238.
[18] W. Takahashi, A convexity in metric spaces and nonexpansive mappings, I, Kodai Math. Sem. Rep. 22 (1970) 142–149.
[19] W.A. Kirk, Krasnoselskii's iteration process in hyperbolic spaces, Numer. Funct. Anal. Optim. 4 (1982) 371–381.
[20] D.W. Boyd, J.S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458–464.
[21] J.L.C. Camargo, An application of a fixed point theorem of D.W. Boyd and J.S.W. Wong, Rev. Mat. Estatist. 6 (1988) 25–29.