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FUNDAMENTAL SIMPLICES WITH OUTER VERTICES FOR HYPERBOLIC GROUPS AND THEIR GROUP EXTENSIONS FOR TRUNCATIONS

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Abstract

There are investigated supergroups of some hyperbolic space groups with simplicial fundamental domain. If the vertices of these simplices are out of the absolute, we can truncate them by polar planes of the vertices and the new polyhedra are fundamental ones of the richer groups. In papers of E. Molnár, I. Prok and J. Szirmai the simplices, investigated here, are collected in families F3, F4 and F6. We have constructed at least one new hyperbolic space group for each truncated simplex in these families.

1 Introduction

Hyperbolic space groups are isometry groups, acting discontinuously on the hyperbolic 3-space with a compact fundamental domain. One possibility to describe them is to look for the fundamental domains of these groups. Face pairing identifications of a given polyhedron give us generators and relations for a space group by the Poincarè Theorem [3, 4, 7].

The simplest fundamental domains are simplices and truncated simplices by the polar planes of their vertices when they lie out of the absolute. In the process of classifying the fundamental simplices, it is determined 64 combinatorially different face pairings of fundamental simplices [19, 20, 10], furthermore 35 solid transitive non-fundamental simplex identifications [10]. I. K. Zhuk [19, 20] has classified Euclidean and hyperbolic fundamental simplices of finite volume up to congruence. Some completing cases are discussed in [6, 9, 15, 16, 17, 18]. An algorithmic procedure is given by E. Molnár and I. Prok [9]. In [10, 12, 13] the authors have summarized all these results, arranging identified simplices into 32 families. Each of them is characterized by the so-called maximal series of simplex tilings. Besides spherical, Euclidean, hyperbolic realizations there exist also other metric

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realizations in 3-dimensional simply connected homogeneous Riemannian spaces, moreover, metrically non-realizable topological simplex tilings occur as well.

If a simplex is hyperbolic with vertices out of the absolute, the simplex is not compact and then it is possible to truncate it with polar planes of the vertices. The new compact polyhedron (here called trunc-simplex) obtained in that way is the fundamental domain of some larger group. It has new triangular faces whose appropriate pairing gives new generators. Dihedral angles around the new edges are $\pi/2$. That means there are four congruent trunc-simplices around them in the fundamental space filling. A trivial group extension, with plane reflections in polar planes of the outer vertices, is always possible. All the other possibilities, to equip new pairings of triangular faces obtained by the truncations, will be considered as well. For that purpose, the stabilizer group of the corresponding vertex figure will be analyzed.

The Poincarè theorem will be recalled in Section 2. Descriptions of the families F3, F4, F6 and their realizations are given in Section 3, by [10, 12, 13]. Supergroups, obtained by trunc-simplices, are given in Section 4 for family F3 and in Section 5 for families F4, F6, by tables. For them it holds the following

Theorem 1. *For the trunc-simplices of the simplices in family F3, all the maximally possible hyperbolic space group series are given in Section 4 by their presentation with generators and relations. For the simplices in families F4 and F6 these group series are given in Tables in Section 5. These will be on the base of inner symmetries of the vertex stabilizer groups (as hyperbolic plane groups). Namely, their normalizers, each preserving the triangulation and orbit structure of the corresponding vertex stabilizer, provide the possible group extensions. We have obtained the new hyperbolic space group series at all.*

The proof will need case-by-case discussions, summarized in the Tables. The typical steps will be illustrated only for Family 3 by simplex T_{42}, and (briefly) by T_{33}. The plane methods of papers [1, 2, 14] can be applied to our spatial situations. The final realization steps – on the base of Coxeter's reflection supergroups, given in the paper [13] – proceed by linear algebra with the reflection simplices as projective coordinate simplices, illustrated here, too, in section 3 (see also [11], e.g. for analogous situation). Then the Mostow rigidity theorem guarantees uniqueness of the metric realizations, because of compactness of the occurring orbifolds.

2 Construction of discontinuously acting isometry groups

Generators and relations for a space group G with a given polyhedron P (a simplex or a trunc-simplex in the considered cases) as a fundamental domain can be obtained by the Poincarè theorem. It is necessary to consider all face pairing identifications of such domains. Those will be isometries, which generate an isometry group G and induce subdivision of vertices and oriented edge segments of P into equivalence classes, such that an edge segment does not contain two G-equivalent points in its interior.

Face pairing identifications are isometries satisfying conditions a)–c). They generate an isometry group G of a space of constant curvature.

a) For each face $f_{g^{-1}}$ of P there is another face f_g and identifying isometry g which maps $f_{g^{-1}}$ onto f_g and P onto P^g, the neighbour of P along f_g.

b) The isometry g^{-1} maps the face f_g onto $f_{g^{-1}}$ and P onto $P^{g^{-1}}$, joining the simplex P along $f_{g^{-1}}$.

c) Each edge segment e_1 from any equivalence class (defined below) is successively surrounded by polyhedra P, $P^{g_1^{-1}}$, $P^{g_2^{-1}g_1^{-1}}$, ..., $P^{g_r^{-1}\cdots g_2^{-1}g_1^{-1}}$, which fill an angular region of measure $2\pi/\nu$, with a natural number ν. An equivalence class consisting of edge segments e_1, e_2, ..., e_r with dihedral angles $\varepsilon(e_1)$, $\varepsilon(e2e_2)$, ..., $\varepsilon(e_r)$, respectively, is defined as follows.

Let us consider an edge segment, say e_1, and choose one of the two faces denoted by $f_{g_1^{-1}}$ whose boundary contains e_1. The isometry g_1 maps e_1 and $f_{g_1^{-1}}$ onto e_2 and f_{g_1}, respectively. There exists exactly one other face $f_{g_2^{-1}}$ with e_2 on its boundary, furthermore the isometry g_2 mapping e_2 and $f_{g_2^{-1}}$ onto e_3 and f_{g_2}, respectively, and so on. We obtain a cycle of isometries g_1, g_2, ..., g_r according to the scheme

$$\left(e_1, f_{g_1^{-1}}\right) \stackrel{g_1}{\to} (e_2, f_{g_1})\,;\left(e_2, f_{g_2^{-1}}\right) \stackrel{g_2}{\to} (e_3, f_{g_2})\,;\ldots;\left(e_r, f_{g_r^{-1}}\right) \stackrel{g_r}{\to} (e_1, f_{g_r}) \qquad (1)$$

where the symbols are not necessarily distinct. More precisely, we have two essentially different cases for the scheme (1).

1. if a plane reflection $m_i = g_i$ occurs then $e_{i+1} = e_i$, and we turn back to e_1, then, say, e_{-1} comes. Furthermore, another plane reflection $m_{-j} = g_{-j}$ shall appear in the cycle. Then each edge segment comes two times in the scheme (1), and the cycle transformation is of the form

$$c = g_1 g_2 \ldots g_r = \left(g_1 \ldots g_{i-1} m_i g_{i-1}^{-1} g_1^{-1}\right)\left(g_{-1}^{-1} g_{-j+1}^{-1} m_{-j} g_{-j+1} g_{-1}\right)$$

2. there is no plane reflection in the cycle; this will be the simpler case. (In dimension 3 we have 5 subcases for the edges at all [7]).

In other words the segment e1 is successively surrounded by polyhedra

$$P, P^{g_1^{-1}}, P^{g_2^{-1}g_1^{-1}}, \ldots, P^{g_r^{-1}\cdots g_2^{-1}g_1^{-1}}$$

which fill an angular region of measure $2\pi/\nu$. In the above case 1. the following holds

$$\varepsilon(e_1) + \cdots + \varepsilon(e_i) + \varepsilon(e_{-1}) + \cdots + \varepsilon(e_{-1+j}) = \pi/\nu. \qquad (2)$$

In case 2. we have

$$\varepsilon(e_1) + \cdots + \varepsilon(e_r) = 2\pi/\nu. \tag{3}$$

Finally, the cycle transformation $c = g_1 g_2 \dots g_r$ belonging to the edge segment class $\{e_1\}$ is a rotation, say, of order ν. Thus we have the cycle relation in both cases

$$(g_1 g_2 \dots g_r)^\nu = 1. \tag{4}$$

Throughout in this paper we shall apply the specified Poincaré theorem:

Theorem 2. *Let P be a polyhedron in a space S^3 of constant curvature and G be the group generated by the face identifications, satisfying conditions a)–c). Then G is a discontinuously acting group on S^3, P is a fundamental domain for G and the cycle relations of type (4) for every equivalence class of edge segments form a complete set of relations for G, if we also add the relations $g_i^2 = 1$ to the occasional involutive generators $g_i = g_i^{-1}$.*

3 Descriptions of the considered families of simplices, proof of the metric existence

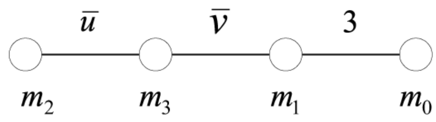
In family F3 there are two series of fundamental simplices. The groups for them are denoted in [10, 12, 13] by $\Gamma_{33}\,(12u, 6v)$, $\Gamma_{42}\,(6u, 3v)$, so the simplices will be denoted here by $T_{33}\,(12u, 6v)$ (Fig.5) and $T_{42}\,(6u, 3v)$ (Fig.1). Vertices A_0, A_1 and A_3 are in one class of equivalence (class a), while A_2 is in another (class b). There are also two classes of equivalence for edges $u : \{A_0A_1, A_0A_3, A_1A_3\}$ and $v : \{A_0A_2, A_1A_2, A_2A_3\}$.

Six series of fundamental simplices in family F4 (Fig.8,9,10) have groups denoted by $\Gamma_{10}\,(2u, 8v, 4w)$, $\Gamma_{17}\,(2u, 4v, 2w)$, $\Gamma_{28}\,(2u, 8v, 4w)$, $\Gamma_{38}\,(2u, 4v, 2w)$, $\Gamma_{54}\,(u, 4v, w)$, $\Gamma_{57}\,(u, 4v, 2w)$ in [10, 12, 13]. There are two classes of vertices $a : \{A_0, A_1\}$ and $b : \{A_2, A_3\}$. For edges there are three classes of equivalence $u : \{A_0A_1\}$, $v : \{A_0A_2, A_0A_3, A_1A_2, A_1A_3\}$, $w : \{A_2A_3\}$.

There are four series of fundamental simplices in family F6 (Fig.11,12) with group notations $\Gamma_6\,(2u, 4v, 4w, 2x)$, $\Gamma_{20}\,(2u, 4v, 4w, x)$, $\Gamma_{24}\,(4u, 2v, 4w, x)$, $\Gamma_{35}\,(2u, 2v, 2w, x)$. There are three classes of equivalence for vertices $a : \{A_0, A_1\}$, $b : \{A_2\}$, $c : \{A_3\}$ and four classes for edges $u : \{A_0A_1\}$, $v : \{A_0A_2, A_1A_2\}$, $w : \{A_0A_3, A_1A_3\}$, $x : \{A_2A_3\}$.

The sum of dihedral angles around edges in the same equivalence class is always of the form $2\pi/\nu$. That is the reason to introduce parameters for each equivalence class of edges. Parameters are denoted by the same letters as the corresponding equivalence classes.

The simplices in family F3 can metrically be derived by [12, 13] from the Coxeter reflection group and simplex denoted here by ${}^{3m}_{6}\Gamma\,(2\bar{u}, \bar{v})$ $(2 \leq \bar{u}, 3 \leq \bar{v})$ with diagram



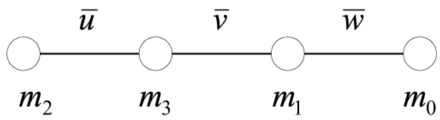
so that e.g. for $T_{33}\,(12u, 6v)$ $2\bar{u} = 12u$, $\bar{v} = 6v$.

That means that this Coxeter group is a supergroup of $\Gamma_{33}\,(12u, 6v)$ by dihedral group extension **3m = *3** of index 6 with 6-times smaller Coxeter simplex, according to the parameters $\bar{u} = 6u$ $\bar{v} = 6v$ ($u \geq 1$, $v \geq 1$ as rotational orders at the corresponding simplex edges). If we know (e.g. by [5, 7, 11, 13]) the linear algebraic construction of Coxeter simplices in projective metric space, then the bigger simplices can also be constructed, now by hyperbolic metric, with the same letter denotation of the parameters.

Analogous arguments hold for $T_{42}\,(6u, 3v)$ with ($u \geq 1$, $v \geq 1$).

Outer vertices A_0, A_1, A_3 and truncations occur iff $\frac{1}{2} + \frac{1}{\bar{u}} + \frac{1}{\bar{v}} < 1$. E.g. for T_{42} holds $\frac{1}{2} + \frac{1}{3u} + \frac{1}{3v} < 1$ (i.e. $\frac{1}{u} + \frac{1}{v} < \frac{3}{2}$). Vertex A_2 is outer iff $\frac{1}{2} + \frac{1}{3} + \frac{1}{\bar{v}} < 1$, i.e. for T_{42}, $\frac{1}{2} + \frac{1}{3} + \frac{1}{3v} < 1$ (for $v > 2$).

The simplices in family F4 can metrically be derived from the Coxeter reflection group ${}^{mm2}_{4}\Gamma_1\,(\bar{u}, 2\bar{v}, \bar{w})$ and simplex ${}^{mm2}_{4}T_1\,(\bar{u}, 2\bar{v}, \bar{w})$ ($3 \leq \bar{u} \leq \bar{w}$, $2 \leq \bar{v}$) with diagram

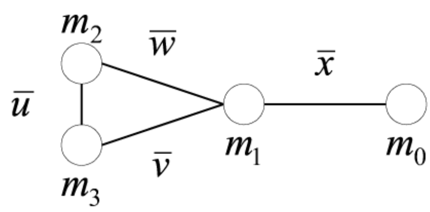


so that this Coxeter group is a supergroup (by inner symmetries **mm2 = *2** of index 4) of the given group, for the corresponding rotational parameters u, v, w.

Again these facts lead to linear algebraic construction in appropriate projective metric space, now with hyperbolic (+,+,+,-) signature. We examine parameters (infinite series) where outer vertices occur by the above equations for vertex figures (hyperbolic fundamental domains for vertex stabilizer groups).

Vertices A_0, A_1 are outer iff $\frac{1}{2}+\frac{1}{\bar{u}}+\frac{1}{\bar{v}} < 1$, while A_2, A_3 are outer iff $\frac{1}{2}+\frac{1}{\bar{u}}+\frac{1}{\bar{w}} < 1$.

Similarly, simplices in family F6 can metrically be derived from the Coxeter reflection group ${}^{m}_{2}\Gamma\,(2\bar{u}, 2\bar{v}, 2\bar{w}, \bar{x})$ ($2 \leq \bar{u}$, $2 \leq \bar{v} \leq \bar{w}$, $3 \leq \bar{x}$) and appropriate simplex, with diagram



Here, vertices A_0, A_1 are outer iff $\frac{1}{\bar{u}} + \frac{1}{\bar{v}} + \frac{1}{\bar{w}} < 1$, A_2 is outer iff $\frac{1}{2} + \frac{1}{\bar{v}} + \frac{1}{\bar{x}} < 1$, A_3 is outer iff $\frac{1}{2} + \frac{1}{\bar{w}} + \frac{1}{\bar{x}} < 1$.

4 The isometry groups by simplices in family F3

4.1 Simplex $T_{42}\,(6u, 3v)$

Face pairing isometries for $T := T_{42}\,(6u, 3v)$ (Fig.1) are

$$r_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_3 & A_2 & A_1 \end{pmatrix}; r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}; z : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_1 & A_2 & A_0 \end{pmatrix}.$$

Relations for the isometry group are obtained by Theorem 2 and the presentation is

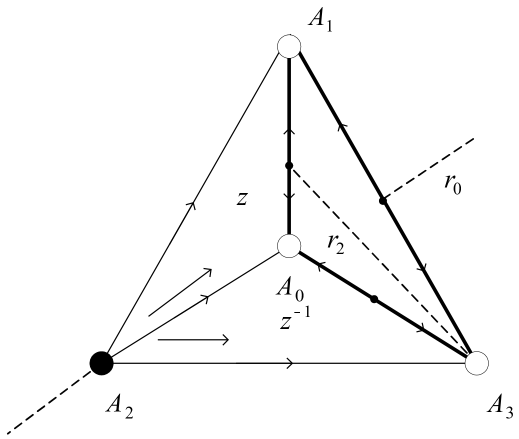


Figure 1: The simplex $T_{42}\,(6u, 3v)$

$$\Gamma_{42}(6u, 3v) = (r_0, r_2, z - r_0^2 = r_2^2 = (z^2 r_0)^v = (r_2 z^{-1} r_2 r_0 r_2 z)^u = 1, 1 \leq u, 1 \leq v).$$

Considering vertex figures on a symbolic 2-dimensional surface (plane) around the vertices, we can glue a fundamental domain for the stabilizer subgroup, e.g. Γ_{A_3} of vertex A_3 and also Γ_{A_2} of vertex A_2. Transformations z and r_0 map vertex A_3 onto A_0 and A_1 respectively, so that the inverse mapping z^{-1} carries the vertex domain T_{A_0} into $T_{A_0}^{z^{-1}}$ the neighbour of T_{A_3} along $f_{z^{-1}}$, and similarly with $T_{A_1}^{r_0^{-1}} = T_{A_1}^{r_0}$ (since $r_0^2 = 1$). We left the letter f from the face symbols in our figures. That means that T_{A_3} and $T_{A_1}^{r_0^{-1}}$ have a joint edge corresponding to the joint face $f_{z^{-1}}$ of simplex T and $f_{z^{-1}}$ of simplex $T^{z^{-1}}$ and similarly, T_{A_3} and $T_{A_1}^{r_0}$ have a joint edge corresponding to $f_{r_0} = f_{r_0}^{-1}$ of T and T^{r_0}. One fundamental domain for $\Gamma(A_3)$ (Fig.2) is

$$P_{A_3} := T_{A_0}^{z^{-1}} \cup T_{A_3} \cup T_{A_1}^{r_0}$$

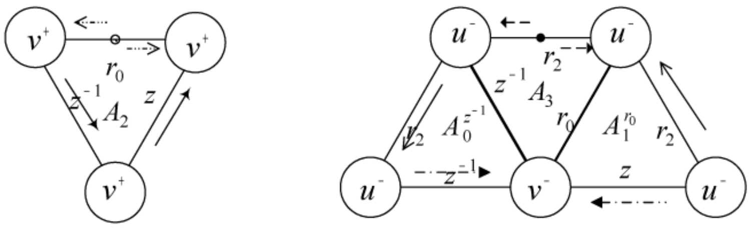


Figure 2: The fundamental domains P_{A_2} and P_{A_3}

and the generators for $\Gamma(A_3)$, obtained from P_{A_3}, are

$$r_2 : f_{r_2} \to f_{r_2}; zr_2r_0 : (f_{r_2})^{z^{-1}} \to (f_{r_2})^{r_0}; z^2r_0 : (f_{z^{-1}})^{z^{-1}} \to (f_z)^{r_0}.$$

In the diagram for P_{A_3} the minus sign in notations u^-, v^- means that edges in these classes are directed to the vertex, the plus sign in diagram for P_{A_2} means the opposite direction. Fundamental domain for P_{A_2} is given in Fig.2 and the generators are

$$r_0 : f_{r_0} \to f_{r_0}; z : f_{z^{-1}} \to f_z.$$

When parameters u, v are such that simplex T is hyperbolic and that the vertices either in the first or in the second equivalence class are out of the absolute (see Sect. 3), it is possible to truncate the simplex by polar planes of these vertices. Then we get a compact trunc-simplex (possibly with 8 faces, as octahedron) denoted by $O_{42} := O$. If we equip O with additional face pairing isometries, it will be a fundamental domain for a group $G_i(O_{42}, 6u, 3v)$ which will be a supergroup of $\Gamma_{42}(6u, 3v)$. A trivial group extension with plane reflections in polar planes of the outer vertices is always possible. For vertices A_0, A_1, A_3 the new relations (Fig.3), which are necessary to add to those of group $\Gamma_{42}(6u, 3v)$ are as follows

$$(\overline{m}_3r_2)^2 = \overline{m}_3z\overline{m}_0z^{-1} = \overline{m}_3r_0\overline{m}_1r_0 = \overline{m}_0r_2\overline{m}_1r_2 = \overline{m}_0z\overline{m}_1z^{-1} = \overline{m}_0^2 = \overline{m}_1^2 = \overline{m}_3^2 = 1$$

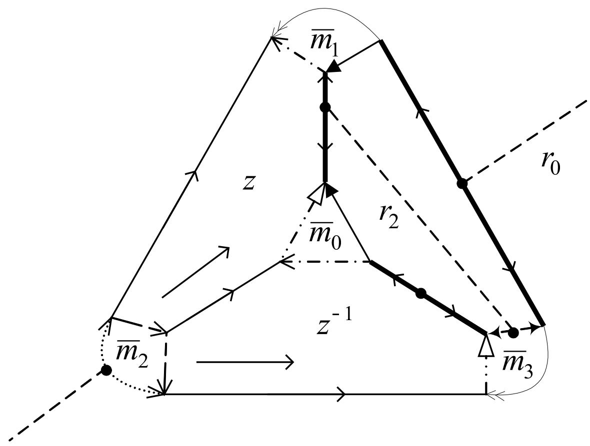


Figure 3: The trunc-simplex O with trivial group extension

where $\overline{m}_i$ denotes the reflection in the polar plane of vertex A_i.

For vertex A_2 these relations with a trivial extension are (Fig.3)

$$(\overline{m}_2 r_0)^2 = \overline{m}_2 z \overline{m}_2 z^{-1} = \overline{m}_2^2 = 1.$$

There is one further possibility to equip the new triangular faces corresponding to outer vertices A_0, A_1, A_3 with face pairing isometries. New additional face pairings of O for vertices A_0, A_1, A_3 have to satisfy the following criteria. Polar plane of A_3 and so stabilizer $\Gamma(A_3)$ will be invariant under these new transformations, fixing A_3, and exchanging the half spaces obtained by the polar plane. Thus, fundamental domain P_{A_3} is divided into two parts, and the new stabilizer of the polar plane will be a supergroup for $\Gamma(A_3)$, namely of index two. Inner symmetries of the P_{A_3}-tiling give us the idea how to introduce the new generators. If $\overline{r}_3$ is the new half-turn mapping the vertex figure T_{A_3} onto itself and $T_{A_1}^{r_0}$ onto $T_{A_0}^{z^{-1}}$, but exchanging the half-spaces, then the new generators for $G_i(O_{42}, 6u, 3v)$ will be $\overline{r}_3$ and $\overline{z} = r_0 \overline{r}_3 z$ and the new relations are (Fig.3, by Poincaré theorem 2)

$$(\overline{r}_3 r_2)^2 = \overline{r}_3 r_0 \overline{z} z^{-1} = \overline{z} r_2 \overline{z} r_2 = (\overline{z} z)^2 = \overline{r}_3^2 = 1.$$

The new generators $\overline{r}_3$ and $\overline{z}$, moreover the new relations can be derived by P_{A_3} and its side pairing above (Fig.2) in standard way. Similarly, for the outer vertex A_2, it is possible to equip a new triangular face with the new half-turn $\overline{r}_2$, so that the new relations are (Fig.4)

$$(\overline{r}_2 r_0)^2 = \overline{r}_2 z \overline{r}_2 z = \overline{r}_2^2 = 1.$$

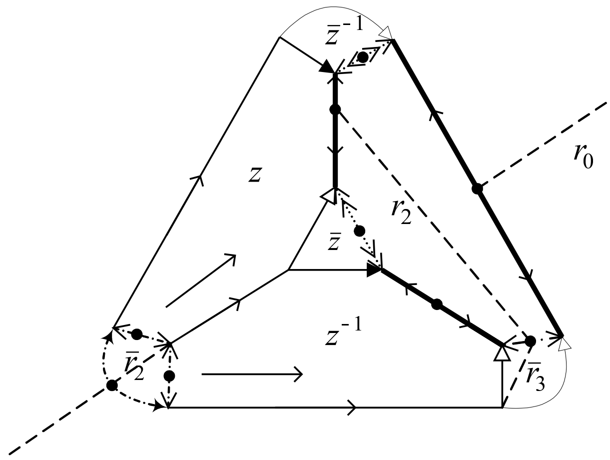


Figure 4: The trunc-simplex O with non-trivial group extension

Remark 1. *Truncations of vertices in different equivalence classes can independently be combined; even there are possibilities to truncate only one or none of classes of equivalence. So there may be 8 possibilities to create supergroups of $\Gamma_{42}(6u, 3v)$ in this way. For all cases of the maximally 4 supergroup series with compact fundamental domains, the notation $G_i(O_{42}, 6u, 3v)$ $i = 1, 2, 3, 4$ can be used, for simplicity.*

4.2 Simplex $T_{33}\,(12u, 6v)$

For $T := T_{33}(12u, 6v)$, the face pairing isometries are (Fig.5):

$$m_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}; r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}; z : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_1 & A_2 & A_0 \end{pmatrix},$$

and the tiling group is

$$\Gamma_{33}(12u, 6v) = (m_0, r_2, z - m_0^2 = r_2^2 = (m_0 r_2 z r_2 z^{-1} r_2 m_0 r_2 z r_2 z^{-1} r_2)^u =$$

$$(m_0 z^{-2} m_0 z^2)^v = 1, 1 \leq u, 1 \leq v).$$

One fundamental domain for the stabilizer group $\Gamma(A_0)$ of the vertex A_0 (Fig.7) is

$$P_{A_0} := T^z_{A_3} \cup T_{A_0} \cup T^{z^{-1}}_{A_1}$$

and the generators are then

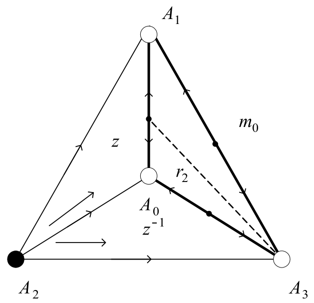


Figure 5: $T_{33}\,(12u, 6v)$

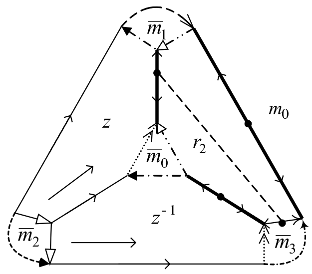


Figure 6: The trunc-simplex O

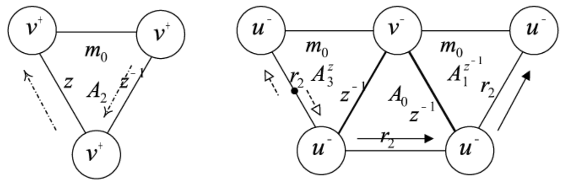


Figure 7: P_{A_2} and P_{A_0}

$$r_2 z^{-1} : f_{r_2} \to (f_{r_2})^{z^{-1}}\,; z^{-1} r_2 z : (f_{r_2})^z \to (f_{r_2})^z\,;$$
$$z m_0 z^{-1} : (f_{m_0})^{z^{-1}} \to (f_{m_0})^{z^{-1}}\,; z^{-1} m_0 z : (f_{m_0})^z \to (f_{m_0})^z\,.$$

The generators for $\Gamma(A_2)$ are

$$m_0 : f_{m_0} \to f_{m_0}; z : f_{z^{-1}} \to f_z.$$

Since parameters u, v are such that the simplex is hyperbolic with outer vertices, then after truncating the simplex by polar planes of such vertices, a new trunc-simplex $O := O_{33}$ may have plane reflections as face pairing isometries to the new faces. For the class of vertices A_0, A_1, A_3, new relations are (Fig.6)

$$\overline{m}_0 r_2 \overline{m}_1 r_2 = (\overline{m}_3 r_2)^2 = \overline{m}_3 z \overline{m}_0 z^{-1} = \overline{m}_0 z \overline{m}_1 z^{-1} = (\overline{m}_1 m_0)^2 = (\overline{m}_3 m_0)^2 =$$
$$\overline{m}_0^2 = \overline{m}_1^2 = \overline{m}_3^2 = 1,$$

and for A_2

$$(\overline{m}_2 m_0)^2 = \overline{m}_2 z \overline{m}_2 z^{-1} = \overline{m}_2^2 = 1.$$

The polyhedron O may also have a half-turn as a new isometry for the face obtained after truncating vertex A_2. Then the new relations are

$$(\overline{r}_2 m_0)^2 = \overline{r}_2 z \overline{r}_2 z = \overline{r}_2^2 = 1$$

For the class of vertices A_0, A_1, A_3, there are no other possibilities for face pairing with isometries, except trivial group extension. That means $G_i(O_{33}, 12u, 6v)$, $i = 1, 2$ are the possible **two** group extensions for $\Gamma_{33}(12u, 6v)$.

5 The isometry groups for simplices in families F4 and F6

For the 6 simplices (Fig.8,9,10) in family F4 and 4 simplices (Fig.11,12) in family F6 our results are arranged in tables. The following data are given:

1. Notation for isometry group of each of the simplices, by [10, 13]; number of group extensions.
2. Face pairing isometries.
3. Presentation of isometry group.
4. New relations for each of the trunc-simplices, if the additional pairing isometries are plane reflections (trivial face pairings). There are arranged in the following way
   - If the original simplex is in family F4, than:
     a) Relations for class of vertices A_0, A_1,
     b) Relations for class of vertices A_2, A_3.
   - If the original simplex is in family F6, than:
     a) Relations for class of vertices A_0, A_1,
     b) Relations for class of vertex A_2,
     c) Relations for class of vertex A_3.
5. One of the fundamental domains for each class of vertices (if there are more then one vertex in the class) and generators for stabilizer group of vertices based on that domain.
6. If the stabilizer groups of vertices have nontrivial normalizing symmetries, there exist non-trivial face pairing isometries for trunc-simplices. There are given relations sorted by classes of vertices as before.

More precisely, for $\Gamma(A_0)$ in cases of simplices Γ_{10} and Γ_{28} there are no other symmetries, while in cases of Γ_{17} and Γ_{38} there are half-turns mapping one vertex figure to another. For fundamental domains of $\Gamma(A_2)$ in cases of Γ_{10}, Γ_{17}, Γ_{28} there are half-turns, respectively, with the same property. Fundamental domain of $\Gamma(A_2)$ in case Γ_{38}, and both $\Gamma(A_0)$ and $\Gamma(A_2)$ in cases Γ_{54} and Γ_{57} have:
I: the half-turn mapping one vertex figure to another;
II: the half-turn mapping each vertex figure to itself;
III: point-reflection mapping one vertex figure to another.
In all cases appearing in family F6, fundamental domains for each class of vertices have one half-turns as additional symmetry.

FAMILY 4

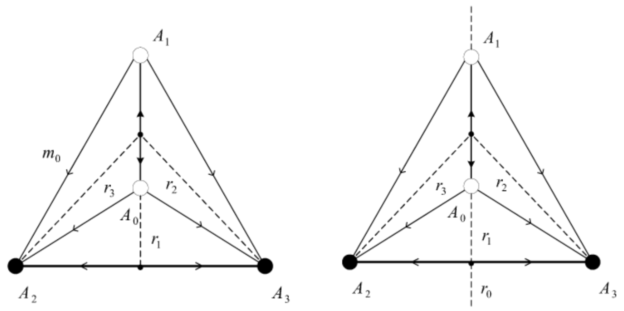


Figure 8: $\Gamma_{10}(2u, 8v, 4w)$ and $\Gamma_{17}(2u, 4v, 2w)$

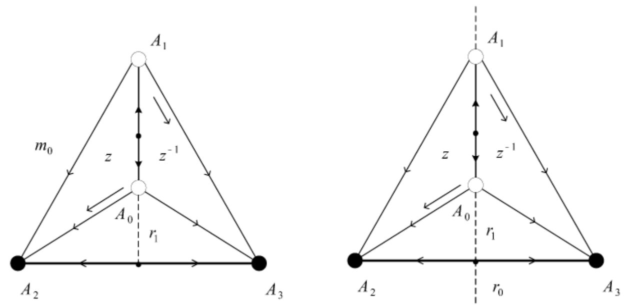


Figure 9: $\Gamma_{28}(2u, 8v, 4w)$ and $\Gamma_{38}(2u, 4v, 2w)$

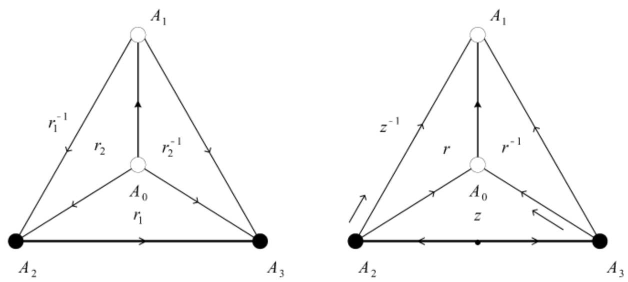


Figure 10: $\Gamma_{54}(u, 4v, w)$ and $\Gamma_{57}(u, 4v, 2w)$

1. $\Gamma_{10}(2u, 8v, 4w)$ (Fig. 5.1); *number of group extensions*: **2**
2. $m_0: \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}$; $r_1: \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; $r_2: \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}$; $r_3: \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_{10}(2u, 8v, 4w) = (m_0, r_1, r_2, r_3 - m_0^2 = r_1^2 = r_2^2 = r_3^2 = (r_2r_3)^u =$ $(m_0 r_3 r_1 r_2 \; m_0 r_2 r_1 r_3)^v = (m_0 r_1 m_0 r_1)^w = 1, 2 \le u, 1 \le v, 1 \le w)$
4. *a)* $(\overline{m}_0 r_1)^2 = \overline{m}_0 r_2 \overline{m}_1 r_2 = \overline{m}_0 r_3 \overline{m}_1 r_3 = (\overline{m}_1 m_0)^2 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $(\overline{m}_2 r_3)^2 = (\overline{m}_3 r_2)^2 = \overline{m}_2 r_1 \overline{m}_3 r_1 = (\overline{m}_2 m_0)^2 = (\overline{m}_3 m_0)^2 = \overline{m}_2^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T^{r_3}_{A_1}$, generators for vert. stab. are $r_1, r_3 m_0 r_3, r_2 r_3$ *b)* f. domain for $\Gamma(A_2)$ is $P_{A_2} := T_{A_2} \cup T^{r_1}_{A_3}$, generators for v.s. are $m_0, r_3, r_1 m_0 r_1, r_1 r_2 r_1$
6. *b)* $(s r_1)^2 = s m_0 s^{-1} m_0 = s r_2 s^{-1} r_3 = 1$

1. $\Gamma_{17}(2u, 4v, 2w)$ (Fig. 5.2); *number of group extensions*: **4**
2. $r_0: \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_3 & A_2 \end{pmatrix}$; $r_1: \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; $r_2: \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}$; $r_3: \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_{17}(2u, 4v, 2w) = (r_0, r_1, r_2, r_3 - r_0^2 = r_1^2 = r_2^2 = r_3^2 = (r_2r_3)^u = (r_1 r_2 \; r_0 \; r_3)^v = (r_0 r_1)^w$ $= 1, 2 \le u \le w, 1 \le v)$
4. *a)* $(\overline{m}_0 r_1)^2 = \overline{m}_0 r_2 \overline{m}_1 r_2 = \overline{m}_0 r_3 \overline{m}_1 r_3 = (\overline{m}_1 r_0)^2 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $(\overline{m}_2 r_3)^2 = (\overline{m}_3 r_2)^2 = \overline{m}_2 r_1 \overline{m}_3 r_1 = \overline{m}_2 r_0 \overline{m}_3 r_0 = \overline{m}_2^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T^{r_2}_{A_1}$, generators for vert. stab. are $r_1, r_3 r_2, r_2 r_0 r_2$ *b)* f. domain for $\Gamma(A_2)$ is $P_{A_2} := T_{A_2} \cup T^{r_1}_{A_3}$, generators for vert. stab. are $r_3, r_0 r_1, r_1 r_2 r_1$
6. *a)* $s_1 r_0 s_1^{-1} r_1 = (s_1 r_2)^2 = (s_1 r_3)^2 = 1$ *b)* $s_2 r_2 s_2^{-1} r_3 = (s_2 r_0)^2 = (s_2 r_1)^2 = 1$

1. $\Gamma_{28}(2u, 8v, 4w)$ (Fig. 5.3); *number of group extensions*: **2**
2. $m_0: \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}$; $r_1: \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; $z: \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_{28}(2u, 8v, 4w) = (m_0, r_1, z - m_0^2 = r_1^2 = z^{2u} = (m_0 z^{-1} r_1 z^{-1} m_0 \; z r_1 z)^v = (m_0 r_1 m_0 r_1)^w$ $= 1, 2 \le u, 1 \le v, 1 \le w)$
4. *a)* $(\overline{m}_0 r_1)^2 = \overline{m}_1 z \overline{m}_0 z^{-1} = \overline{m}_0 z \overline{m}_1 z^{-1} = (\overline{m}_1 m_0)^2 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $\overline{m}_2 r_1 \overline{m}_3 r_1 = (\overline{m}_2 m_0)^2 = (\overline{m}_3 m_0)^2 = \overline{m}_3 z \overline{m}_2 z^{-1} = \overline{m}_2^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T^{z}_{A_1}$, generators for vert. stab. are $r_1, z^2, z^{-1} m_0 z$ *b)* f. domain for $\Gamma(A_2)$ is $P_{A_2} := T_{A_2} \cup T^{r_1}_{A_3}$, generators for vert. stab. are $m_0, r_1 z, r_1 m_0 r_1$
6. *b)* $s m_0 s^{-1} m_0 = (s r_1)^2 = s z s z = 1$

1. **$\Gamma_{38}(2u, 4v, 2w)$** (Fig. 5.4); *number of group extensions*: **8**
2. $r_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_3 & A_2 \end{pmatrix}$; $r_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; $z : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_{38}(2u, 4v, 2w) = (r_0, r_1, z - r_0^2 = r_1^2 = z^{2u} = (r_1 z r_0 z)^v = (r_0 r_1)^w = 1, 2 \leq u \leq w, 1 \leq v)$
4. *a)* $(\overline{m}_0 r_1)^2 = (\overline{m}_1 r_0)^2 = \overline{m}_1 z \overline{m}_0 z^{-1} = \overline{m}_0 z \overline{m}_1 z^{-1} = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $\overline{m}_2 r_1 \overline{m}_3 r_1 = \overline{m}_2 r_0 \overline{m}_3 r_0 = \overline{m}_3 z \overline{m}_2 z^{-1} = \overline{m}_2^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T_{A_1}^{z^{-1}}$, generators for vert. stab. are $r_1, z^2, z r_0 z^{-1}$ *b)* f. domain for $\Gamma(A_2)$ is $P_{A_2} := T_{A_2} \cup T_{A_3}^{z}$, generators for vert. stab. are $r_0 z, r_1 z$
6. *a)* $\overline{z} r_0 \overline{z}^{-1} r_1 = (\overline{z} z)^2 = (\overline{z} z^{-1})^2 = 1$ *b)* I: $(\overline{z}_1 z)^2 = \overline{z}_1 r_0 \overline{z}_1 r_1 = 1$; II: $h_2 r_1 h_3 r_0 = h_2 z^{-1} h_3 z = 1$; III: $s z s z = (s r_0)^2 = (s r_1)^2 = 1$

1. **$\Gamma_{54}(u, 4v, w)$** (Fig. 5.5); *number of group extensions*: **16**
2. $r_1 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}$; $r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_2 \end{pmatrix}$.
3. $\Gamma_{54}(u, 4v, w) = (r_1, r_2 - r_2^u = (r_1 r_2 r_1^{-1} r_2^{-1})^v = r_1^w = 1, 3 \leq u \leq w, 1 \leq v)$
4. *a)* $\overline{m}_1 r_1 \overline{m}_0 r_1^{-1} = \overline{m}_0 r_2 \overline{m}_0 r_2^{-1} = \overline{m}_1 r_2 \overline{m}_1 r_2^{-1} = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $\overline{m}_2 r_1 \overline{m}_2 r_1^{-1} = \overline{m}_3 r_1 \overline{m}_3 r_1^{-1} = \overline{m}_3 r_2 \overline{m}_2 r_2^{-1} = \overline{m}_2^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T_{A_1}^{r_1}$, generators for vert. stab. are $r_2, r_1^{-1} r_2 r_1$ *b)* f. domain for $\Gamma(A_2)$ is $P_{A_2} := T_{A_2} \cup T_{A_3}^{r_2}$, generators for vert. stab. are $r_1, r_2^{-1} r_1 r_2$
6. *a)* I: $(s_1 r_1)^2 = s_1 r_2 s_1^{-1} r_2^{-1} = 1$; II: $r_1 h_0 r_1^{-1} = (r_2 h_0)^2 = (r_2 h_1)^2 = 1$; III: $z_1 r_2 z_1^{-1} r_2 = z_1 r_1 z_1 r_1 = 1$ *b)* I: $s_2 r_1 s_2^{-1} r_1^{-1} = (s_2 r_2)^2 = 1$; II: $r_2 h_2 r_2^{-1} h_3 = (r_1 h_2)^2 = (r_1 h_3)^2 = 1$; III: $z_2 r_1 z_2^{-1} r_1 = z_2 r_2 z_2 r_2 = 1$

1. **$\Gamma_{57}(u, 4v, 2w)$** (Fig. 5.6); *number of group extensions*: **16**
2. $z : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_3 & A_2 \end{pmatrix}$; $r : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_2 \end{pmatrix}$
3. $\Gamma_{57}(u, 4v, 2w) = (r, z - r^u = (r z^{-1} r z)^v = z^{2w} = 1, 3 \leq u, 1 \leq v, 2 \leq w)$
4. *a)* $\overline{m}_0 r \overline{m}_0 r^{-1} = \overline{m}_1 r \overline{m}_1 r^{-1} = \overline{m}_1 z \overline{m}_0 z^{-1} = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $\overline{m}_3 r \overline{m}_2 r^{-1} = \overline{m}_2 z \overline{m}_3 z^{-1} = \overline{m}_3 z \overline{m}_2 z^{-1} = \overline{m}_2^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T_{A_1}^{z}$, generators for vert. stab. are $r, z^{-1} r z$ *b)* f. domain for $\Gamma(A_2)$ is $P_{A_2} := T_{A_2} \cup T_{A_3}^{r}$, generators for vert. stab. are $zr, z^{-1} r$
6. *a* I: $\overline{z}_1 r \overline{z}_1^{-1} r^{-1} = (\overline{z}_1 z)^2 = 1$; II: $(h_0 r)^2 = (h_1 r)^2 = h_1 z h_0 z^{-1} = 1$; III: $s_1 r s_1^{-1} r^{-1} = s_1 z s_1 z = 1$ *b)* I: $s_2 z^{-1} s_2 z = (s_2 r)^2 = 1$; II: $h_2 z h_3 z = h_3 r h_2 r^{-1} = 1$; III: $(\overline{z}_2 z)^2 = (\overline{z}_2 z^{-1})^2 = \overline{z}_2 r \overline{z}_2 r = 1$

FAMILY 6

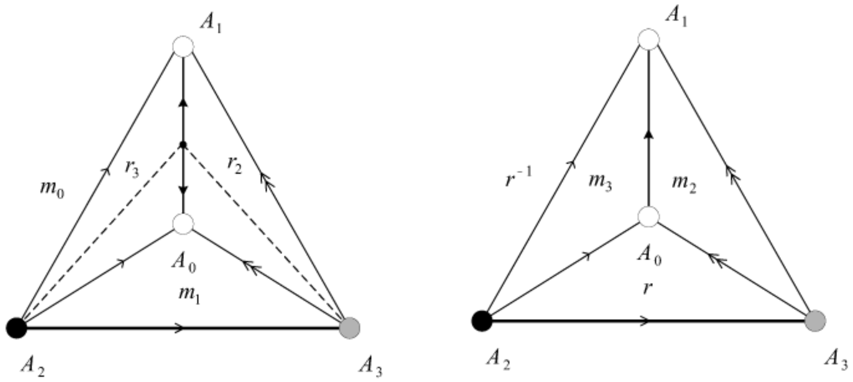


Figure 11: $\Gamma_6(2u, 4v, 4w, 2x)$ and $\Gamma_{20}(2u, 4v, 4w, x)$

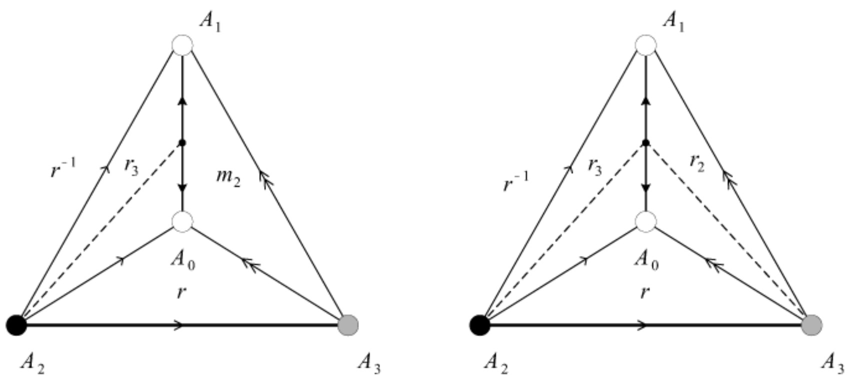


Figure 12: $\Gamma_{24}(4u, 2v, 4w, x)$ and $\Gamma_{35}(2u, 2v, 2w, x)$

1. $\mathbf{\Gamma_6(2u, 4v, 4w, 2x)}$ (Fig. 5.7); *number of group extensions*: **8**
2. $m_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 \end{pmatrix}; m_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}; r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}; r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_6(2u, 4v, 4w, 2x) = (m_0, m_1, r_2, r_3 - m_0^2 = m_1^2 = r_2^2 = r_3^2 = (r_2 r_3)^u = (m_0\, r_3\, m_1\, r_3)^v =$ $(m_0\, r_2\, m_1\, r_2)^w = (m_0\, m_1)^x = 1,\ 2 \le u,\ 1 \le v \le w,\ 2 \le x)$
4. *a)* $(\overline{m}_0 m_1)^2 = \overline{m}_0 r_2 \overline{m}_1 r_2 = \overline{m}_0 r_3 \overline{m}_1 r_3 = (\overline{m}_1 m_0)^2 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $(\overline{m}_2 r_3)^2 = (\overline{m}_2 m_1)^2 = (\overline{m}_2 m_0)^2 = \overline{m}_2^2 = 1$ *c)* $(\overline{m}_3 r_2)^2 = (\overline{m}_3 m_1)^2 = (\overline{m}_3 m_0)^2 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_1)$ is $P_{A_1} := T_{A_1} \cup T_{A_0}^{r_2}$, generators for vert.stab. are $m_0, r_3 r_2, r_2 m_1 r_2$ *b)* for $\Gamma(A_2)$ generators for vert. stab. are m_0, m_1, r_3 *c)* for $\Gamma(A_3)$ generators for vert. stab. are m_0, m_1, r_2
6. *a)* $(s r_2)^2 = (s r_3)^2 = s m_0 s^{-1} m_0 = 1$ *b)* $(r_3 h_2)^2 = m_0 h_2 m_1 h_2 = h_2^2 = 1$ *c)* $(r_2 h_3)^2 = m_0 h_3 m_1 h_3 = h_3^2 = 1$

1. $\mathbf{\Gamma_{20}(2u, 4v, 4w, x)}$ (Fig. 5.8); *number of group extensions*: **8**
2. $r : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix};\ m_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_3 \end{pmatrix};\ m_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_1 & A_2 \end{pmatrix}$
3. $\Gamma_{20}(2u, 4v, 4w, x) = (m_2, m_3, r - m_2^2 = m_3^2 = (m_2 m_3)^u = (m_3\, r^{-1}\, m_3\, r)^v = (m_2\, r^{-1}\, m_2\, r)^w$ $= r^x = 1,\ 2 \le u,\ 1 \le v \le w,\ 3 \le x)$
4. *a)* $(\overline{m}_1 m_3)^2 = (\overline{m}_1 m_2)^2 = (\overline{m}_0 m_3)^2 = (\overline{m}_0 m_2)^2 = r \overline{m}_0 r^{-1} \overline{m}_1 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $(\overline{m}_2 m_3)^2 = r \overline{m}_2 r^{-1} \overline{m}_2 = \overline{m}_2^2 = 1$ *c)* $(\overline{m}_3 m_2)^2 = r \overline{m}_3 r^{-1} \overline{m}_3 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \cup T_{A_1}^{r}$, generators for v.s.are $m_2, m_3, r^{-1} m_2 r, r^{-1} m_3 r$ *b)* for $\Gamma(A_2)$ generators for vert. stab. are r, m_3 *c)* for $\Gamma(A_3)$ generators for vert. stab. are r, m_2
6. *a)* $s m_2 s^{-1} m_2 = s m_3 s^{-1} m_3 = (r s)^2 = 1$ *b)* $(h_2 m_3)^2 = (r h_2)^2 = h_2^2 = 1$ *c)* $(h_3 m_2)^2 = (r h_3)^2 = h_3^2 = 1$

1. $\Gamma_{24}(4u, 2v, 4w, x)$ (Fig. 5.9); *number of group extensions*: **8**
2. $r : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}$; $m_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_0 & A_1 & A_3 \end{pmatrix}$; $r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_{24}(4u, 2v, 4w, x) = (m_2, r_3, r - m_2^2 = r_3^2 = (m_2r_3m_2r_3)^u = (r\, r_3)^v = (m_2\, r^{-1}\, m_2\, r)^w =$ $r^x = 1, 1 \le u, 2 \le v, 1 \le w, 3 \le x)$
4. *a)* $r\overline{m}_0r^{-1}\overline{m}_1 = r_3\overline{m}_0r_3\overline{m}_1 = (\overline{m}_0m_2)^2 = (\overline{m}_1m_2)^2 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $(\overline{m}_2r_3)^2 = r\overline{m}_2r^{-1}\overline{m}_2 = \overline{m}_2^2 = 1$ *c)* $(m_2\overline{m}_3)^2 = r\overline{m}_3r^{-1}\overline{m}_3 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \bigcup T^r_{A_1}$, generators for vert.stab. are m_2, $r^{-1}r_3$, $r^{-1}m_2r$ *b)* for $\Gamma(A_2)$ generators for vert. stab. are r, r_3 *c)* for $\Gamma(A_3)$ generators for vert. stab. are r, m_2
6. *a)* $(s\,r_3)^2 = s\,m_2s^{-1}m_2 = (s\,r)^2 = 1$ *b)* $(h_2r_3)^2 = (r\,h_2)^2 = h_2^2 = 1$ *c)* $(h_3m_2)^2 = (r\,h_3)^2 = h_3^2 = 1$

1. $\Gamma_{38}(2u, 2v, 2w, x)$ (Fig. 5.10); *number of group extensions*: **8**
2. $r : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_0 & A_2 & A_3 \end{pmatrix}$; $r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}$; $r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$
3. $\Gamma_{38}(2u, 2v, 2w, x) = (r_2, r_3, r - r_2^2 = r_3^2 = (r_2r_3)^u = (r\, r_3)^v = (r\, r_2)^w = r^x = 1, 2 \le u,$ $2 \le v \le w, 3 \le x)$
4. *a)* $r\overline{m}_0r^{-1}\overline{m}_1 = \overline{m}_0r_3\overline{m}_1r_3 = \overline{m}_0r_2\overline{m}_1r_2 = \overline{m}_0^2 = \overline{m}_1^2 = 1$ *b)* $(\overline{m}_2r_3)^2 = r\overline{m}_2r^{-1}\overline{m}_2 = \overline{m}_2^2 = 1$ *c)* $(\overline{m}_3r_2)^2 = r\overline{m}_3r^{-1}\overline{m}_3 = \overline{m}_3^2 = 1$
5. *a)* f. domain for $\Gamma(A_0)$ is $P_{A_0} := T_{A_0} \bigcup T^r_{A_1}$, generators for vert. stab. are r_2r, r_3r *b)* for $\Gamma(A_2)$ generators for vert. stab. are r, r_3 *c)* for $\Gamma(A_3)$ generators for vert. stab. are r, r_2
6. *a)* $(s\,r_3)^2 = (s\,r_2)^2 = (r\,s)^2 = 1$ *b)* $(h_2r_3)^2 = (r\,h_2)^2 = h_2^2 = 1$ *c)* $(h_3r_2)^2 = (r\,h_3)^2 = h_3^2 = 1$

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