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Fuzzy common fixed point theorems for generalized contractive mappings

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ABSTRACT

We prove the existence of a fuzzy common fixed point of two mappings satisfying a generalized contractive condition. Our results provide extensions as well as substantial improvements of several well known results from the existing literature.

1. Introduction and preliminaries

Let X be a space of points with generic element of X denoted by x and $I = [0, 1]$. A fuzzy subset of X is characterized by a membership function which associates with each element in X a real number in the interval I. Let (X, d) be a metric linear space and A be a fuzzy set in X characterized by a membership function A. The α-level set of A, denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if } \alpha \in (0, 1]$$
$$A_0 = \overline{\{x : A(x) > 0\}}$$

where $\overline{B}$ denotes the closure of the non-fuzzy set B.

A fuzzy set A in a metric linear space is said to be an approximate quantity if and only if A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup_{x\in X} A(x) = 1$. We denote by $W(X)$ the family of all approximate quantities in X.

Suppose that $A, B \in W(X)$; then A is said to be more accurate than B, denoted by $A \subset B$, if and only if $A(x) \leq B(x)$ for each x in X, where B denotes the membership function of B. For $x \in X$, we write $\{x\}$ for the characteristic function of the ordinary subset $\{x\}$ of X. We define $W^0(X) = \{\{x\} : x \in X\}$.

For $\alpha \in (0, 1]$, the fuzzy point $(x)_\alpha$ of X is the fuzzy set of X given by $x_\alpha(x) = \alpha$ and $\alpha \neq x$.

Let I^X be the collection of all fuzzy subsets in X and $W(X)$ be a subcollection of all approximate quantities. For $A, B \in W(X)$, $\alpha \in [0, 1]$, define

$$p_\alpha(A, B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y),$$
$$p(A, B) = \sup_\alpha P_\alpha(A, B),$$

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$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$
$$D(A, B) = \sup_\alpha D_\alpha(A, B),$$

where H is the Hausdorff metric H induced by the metric d. We note that P_α is non-decreasing function of α and D is a metric on $W(X)$.

Suppose that $\alpha \in [0, 1]$; then the family $W_\alpha(X)$ is given by $\{A \in I^X : A_\alpha$ is non-empty convex and compact$\}$.

Let X be an arbitrary set, Y be a metric linear space. A mapping T is called a fuzzy mapping if T is a mapping from X into $W(Y)$, that is, $Tx \in W(Y)$ for each x in X. Thus if we characterize a fuzzy set Tx in a metric linear space Y by a membership function Tx, then $Tx(y)$ is the grade of membership of y in Tx. Therefore a fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $Tx(y)$.

A fuzzy point x_α in X is called a fixed fuzzy point of the fuzzy mapping T if $x_\alpha \subset Tx$ [1]. If $\{x\} \subset Tx$, then x is a fixed point of T.

The following lemmas are needed in the sequel.

Lemma 1 (*Heilpern [2]*). *Let (X, d) be a metric space, with $x, y \in X$ and $A, B \in W(X)$;*

(1) *if $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$;*
(2) *$p_\alpha(x, A) \le d(x, y) + p_\alpha(y, B)$;*
(3) *if $x_\alpha \subset A$, then $p_\alpha(x, B) \le D_\alpha(A, B)$.*

Lemma 2 (*Lee and Cho [3]*). *Let (X, d) be a complete metric space and T be a fuzzy mapping from X into $W(X)$ with $x_0 \in X$. Then there exists an $x_1 \in X$ such that $\{x_1\} \subset Tx_0$.*

Zadeh [4] introduced the concept of a fuzzy set which motivated a lot of mathematical activity on the generalization of the notion of a fuzzy set. Heilpern [2] introduced the concept of a fuzzy mappings in a metric linear space and proved a fixed point theorem for fuzzy contraction mapping which is the generalization of a fixed point theorem for multi-valued mappings of Nadler [5]. Estruch and Vidal [1] proved a fixed point theorem for fuzzy contraction mappings in a complete metric space which in turn generalized the Heilpern fixed point theorem. A further generalization of the result given in [1] was proved in [6]. Recently Dutta and Choudhury [7] gave a generalization of the Banach contraction principle, which in turn generalizes Theorem 1 of [8] and the corresponding result of [9]. Very recently Altun et al. [10] proved fixed point theorems in the framework of ordered cone metric spaces. Bose and Shani [11] extended the result of Heilpern to pair of mappings. Sahin et al. [12] also obtained some common fixed point theorems for fuzzy mappings in quasi-pseudo-metric spaces. Recently Azam and Beg [13] proved a common fixed point theorem for mappings which satisfy an Alber and Guerre-Delabriere type contractive condition.

The aim of this work is to establish the existence of a common fuzzy fixed point of generalized contractive mappings without employing any commutativity condition. Our result generalizes, improves and extends many known results from related literature [13,14,6].

2. Main results

We begin with the following result:

Theorem 3. *Let X be a complete metric space, and $T_1, T_2 : X \longrightarrow W_\alpha(X)$ be two fuzzy mappings on X. Suppose that there exists a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $\varphi(t) < t$ for all $t > 0$ and $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t \ge 0$ such that following inequality holds:*

$$D_\alpha(T_1x, T_2y) \le \varphi(M(x, y)) + L \min\{p_\alpha(x, T_1x), p_\alpha(y, T_2y), p_\alpha(x, T_2y), p_\alpha(y, T_1x)\} \quad (1)$$

for all $x, y \in X$, where $L \ge 0$ and

$$M(x, y) = \max\left\{d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{p_\alpha(x, T_2y) + p_\alpha(y, T_1x)}{2}\right\}.$$

Then there exists a point x in X such that $x_\alpha \subset T_1x$ and $x_\alpha \subset T_2x$.

Proof. Let x_0 be in X. By Lemma 2, there exists x_1 in X such that $\{x_1\} \subset T_1x_0$ which implies that

$$p_\alpha(x_1, T_1x_0) = 0 \quad \text{for each } \alpha \text{ in } [0, 1],$$

which is possible if and only if $x_1 \in (T_1x_0)_\alpha$. Since $(T_2x_1)_\alpha$ is a non-empty compact subset of X, there exists $x_2 \in (T_2x_1)_\alpha$ such that

$$d(x_1, x_2) = p_\alpha(x_1, T_2x_1) \le D_\alpha(T_1x_0, T_2x_1).$$

Continuing this process, one obtains a sequence $\{x_n\}$ in X such that $x_{2n+1} \in (T_1x_{2n})_\alpha$ and $x_{2n+2} \in (T_2x_{2n+1})_\alpha$ for all $n \geq 0$ and $d(x_{2n+1}, x_{2n+2}) \leq D_\alpha(T_1x_{2n}, T_2x_{2n+1})$. Taking x_{2n} for x and x_{2n+1} for y in the inequality (1), it follows that

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq D_\alpha(T_1x_{2n}, T_2x_{2n+1}) \\ &\leq \varphi(M(x_{2n}, x_{2n+1})) + L\min\{p_\alpha(x_{2n}, T_1x_{2n}), p_\alpha(x_{2n+1}, T_2x_{2n+1}), p_\alpha(x_{2n}, T_2x_{2n+1}), p_\alpha(x_{2n+1}, T_1x_{2n})\}, \end{aligned}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\left\{d(x_{2n}, x_{2n+1}), p_\alpha(x_{2n}, T_1x_{2n}), p_\alpha(x_{2n+1}, T_2x_{2n+1}), \frac{p_\alpha(x_{2n}, T_2x_{2n+1}) + p_\alpha(x_{2n+1}, T_1x_{2n})}{2}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), p_\alpha(x_{2n}, T_1x_{2n}), p_\alpha(x_{2n+1}, Tx_{2n+1}), \frac{p_\alpha(x_{2n}, T_2x_{2n+1})}{2}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\right\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \end{aligned}$$

which further implies that

$$d(x_{2n+1}, x_{2n+2}) \leq \varphi(\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}).$$

Now if

$$d(x_{2n+1}, x_{2n+2}) > d(x_{2n}, x_{2n+1}),$$

for some n, then we have

$$d(x_{2n+1}, x_{2n+2}) \leq \varphi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

a contradiction. Therefore $d(x_{2n+1}, x_{2n+2}) \leq \varphi(d(x_{2n}, x_{2n+1}))$. Similarly it can be shown that $d(x_{2n+3}, x_{2n+2}) \leq \varphi(d(x_{2n+2}, x_{2n+1}))$. Therefore, for all n,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq D_\alpha(T_1x_{n-1}, T_2x_n) \leq \varphi(d(x_{n-1}, x_n)) \\ &= \varphi(p_\alpha(x_{n-1}, T_2x_{n-1})) \\ &\leq \varphi(D_\alpha(T_1x_{n-2}, T_2x_{n-1})) \\ &\leq \varphi(\varphi(d(x_{n-1}, x_{n-2}))) \\ &\leq \cdots \leq \varphi^n(d(x_0, x_1)). \end{aligned}$$

Hence

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq \varphi^n(d(x_0, x_1)) + \cdots + \varphi^{n+m-1}(d(x_0, x_1)) \\ &= \sum_{k=n}^{n+m-1} \varphi^k(d(x_0, x_1)). \end{aligned}$$

Since $\sum_{n=0}^{\infty} \varphi^n(d(x_0, x_1)) < \infty$, $\{x_n\}$ is a Cauchy sequence in X, and from the completeness of X, it follows that $x_n \longrightarrow x \in X$. Now, we claim that $p_\alpha(x, T_2x) = 0$ for each $\alpha \in [0, 1]$. If not, then for some α^* in $[0, 1]$, we have $p_{\alpha^*}(x, T_2x) > 0$. Consider

$$\begin{aligned} p_{\alpha^*}(x, T_2x) &\leq d(x, x_{2n+1}) + p_{\alpha^*}(x_{2n+1}, T_2x) \\ &\leq d(x, x_{n+1}) + D_{\alpha^*}(T_1x_{2n}, T_2x) \\ &\leq d(x, x_{n+1}) + \max\left\{d(x_{2n}, x), p_{\alpha^*}(x_{2n}, T_1x_{2n}), p_{\alpha^*}(x, T_2x), \frac{p_{\alpha^*}(x_{2n}, T_2x) + p_{\alpha^*}(x, T_1x_{2n})}{2}\right\} \\ &\quad + L\min\{p_{\alpha^*}(x_{2n}, T_1x_{2n}), p_{\alpha^*}(x, T_2x), p_{\alpha^*}(x_{2n}, T_2x), p_{\alpha^*}(x, T_1x_{2n})\} \\ &= d(x, x_{n+1}) + \max\left\{d(x_{2n}, x), d(x_n, x_{2n+1}), p_{\alpha^*}(x, T_2x), \frac{p_{\alpha^*}(x_{2n}, T_2x) + d(x, x_{2n+1})}{2}\right\} \\ &\quad + L\min\{d(x_{2n}, x_{2n+1}), p_{\alpha^*}(x, T_2x), p_{\alpha^*}(x_{2n}, T_2x), p_{\alpha^*}(x, T_1x_{2n})\} \end{aligned}$$

which on taking the limit $n \to \infty$ gives

$$p_{\alpha^*}(x, T_2x) \leq \varphi(p_{\alpha^*}(x, T_2x)) < p_{\alpha^*}(x, T_2x),$$

a contradiction. Hence $p_{\alpha^*}(x, T_2x) = 0$. Therefore $x_\alpha \subset T_2x$. Similarly $x_\alpha \subset T_1x$. $\square$

If in above theorem, we take $\varphi(t) = \theta t$ where $\theta \in (0, 1)$, then we have the following theorem.

Corollary 4. *Let X be a complete metric space, and $T_1, T_2 : X \longrightarrow W_\alpha(X)$ be two fuzzy mapping on X. Suppose that*

$$D_\alpha(T_1x, T_2y) \leq \theta M(x, y) + L \min\{p_\alpha(x, T_1x), p_\alpha(y, T_2y), p_\alpha(x, T_2y), p_\alpha(y, T_1x)\}$$

for all $x, y \in X$, where $L \geq 0$, $\theta \in (0, 1)$, and

$$M(x, y) = \max\left\{d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{p_\alpha(x, T_2y) + p_\alpha(y, T_1x)}{2}\right\}.$$

Then there exists a point x in X such that $x_\alpha \subset T_1x$ and $x_\alpha \subset T_2x$.

If in the above corollary we take $L = 0$ and $T_1 = T_2$, then we have the following result.

Corollary 5. *Let X be a complete metric space. Suppose that a fuzzy mapping $T : X \longrightarrow W_\alpha(X)$ satisfies*

$$D_\alpha(Tx, Ty) \leq \theta \max\left\{d(x, y), p_\alpha(x, Tx), p_\alpha(y, Ty), \frac{p_\alpha(x, Ty) + p_\alpha(y, Tx)}{2}\right\}$$

for all $x, y \in X$, where $\theta \in (0, 1)$. Then there exists x in X such that x_α is a fixed fuzzy point of T.

Corollary 5 is a fuzzy extension of the fixed point theorem given by Berinde [14].

Corollary 6 (*Heilpern [2], Theorem 3.1*)**.** *Let X be a complete metric space, $T : X \longrightarrow W_\alpha(X)$ be a fuzzy mapping on X satisfying*

$$D_\alpha(Tx, Ty) \leq \theta d(x, y)$$

for all $x, y \in X$, where $\theta \in (0, 1)$. Then there exists an x in X such that x_α is a fixed fuzzy point of T.

Theorem 7. *Let X be a complete metric space. Suppose that $T_1, T_2 : X \longrightarrow W_\alpha(X)$ are two fuzzy mappings on X satisfying*

$$D_\alpha(T_1x, T_2y) \leq d(x, y) - \phi(d(x, y)) \tag{2}$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and monotone non-decreasing function with $\phi(0) = 0$. Then there exists a point x in X such that $x_\alpha \subset T_1x$ and $x_\alpha \subset T_2x$.

Proof. Let x_0 be in X. By Lemma 2, there exists x_1 in X such that $\{x_1\} \subset T_1(x_0)$ which implies that

$$p_\alpha(x_1, T_1x_0) = 0 \quad \text{for each } \alpha \text{ in } [0, 1],$$

which is possible if and only if $x_1 \in (T_1x_0)_\alpha$. $(T_2x_1)_\alpha$ is a non-empty compact subset of X; therefore there exists $x_2 \in (T_2x_1)_\alpha$ such that

$$d(x_1, x_2) = p_\alpha(x_1, T_2x_1) \leq D_\alpha(T_1x_0, T_2x_1).$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_{2n+1} \in (T_1(x_{2n}))_\alpha$ and $x_{2n+2} \in (T_2(x_{2n+1}))_\alpha$ for all $n \geq 0$ and $d(x_{2n+1}, x_{2n+2}) \leq D_\alpha(T_1x_{2n}, T_2x_{2n+1})$. Thus by taking x_{2n} for x and x_{2n+1} for y in the inequality (2), it follows that

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\leq D_\alpha(T_1x_{2n}, T_2x_{2n+1}) \\ &\leq d(x_{2n}, x_{2n+1}) - \phi(d(x_{2n}, x_{2n+1})). \end{aligned}$$

Similarly

$$d(x_{2n+3}, x_{2n+2}) \leq d(x_{2n+2}, x_{2n+1}) - \phi(d(x_{2n+2}, x_{2n+1})).$$

Therefore, for all n,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq D_\alpha(T_1x_{n-1}, T_2x_n) \\ &\leq d(x_{n-1}, x_n) - \phi(d(x_{n-1}, x_n)). \end{aligned}$$

Thus, we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$$

which shows that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore converges to a real number x. We show that $x = 0$. If not, then on taking $n \to \infty$,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) - \phi(d(x_{n-1}, x_n))$$

gives that

$$x \leq x - \phi(x) < x,$$

a contradiction. Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Following arguments similar to those given in [15] and [16], it can be shown that $\{x_n\}$ is a Cauchy sequence in X. It follows from the completeness of X that $x_n \longrightarrow x \in X$. Now, we claim that $p_\alpha(x, T_2x) = 0$ for each $\alpha \in [0, 1]$. For this, consider

$$\begin{aligned} p_\alpha(x, T_2x) &\le d(x, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2x) \\ &\le d(x, x_{n+1}) + D_\alpha(T_1x_{2n}, T_2x) \\ &\le d(x, x_{n+1}) + d(x_{2n}, x) - \phi(d(x_{2n}, x)) \end{aligned}$$

which on taking the limit $n \to \infty$ implies that

$$p_\alpha(x, T_2x) \le 0$$

a contradiction. Hence $p_\alpha(x, T_2x) = 0$. Therefore $x_\alpha \subset T_2x$. Similarly, $x_\alpha \subset T_1x$. □

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