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# Subharmonicity of the modulus of quasiregular harmonic mappings

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ABSTRACT

In this note we determine all numbers $q \in \mathbf{R}$ such that $|u|^q$ is a subharmonic function, provided that $u$ is a $K$-quasiregular harmonic mappings in an open subset $\Omega$ of the Euclidean space $\mathbf{R}^n$.



## 1. Introduction

By $|\cdot|$ we denote the Euclidean norm in $\mathbf{R}^n$ and let $\Omega$ be a region in $\mathbf{R}^n$. In this paper we consider $K$-quasiregular harmonic mappings, where $K \geqslant 1$. We recall that a harmonic mapping $u(x) = (u_1(x), \dots, u_n(x)) : \Omega \to \mathbf{R}^n$ with formal differential matrix

$$Du(x) = \{\partial_i u_j(x)\}_{i,j=1}^n$$

is $K$-quasiregular if

$$K^{-1}|Du(x)|^n \leqslant J_u(x) \leqslant Kl(Du(x))^n, \quad \text{for all } x \in \Omega, \tag{1.1}$$

where $J_u$ is the Jacobian of $u$ at $x$,

$$|Du| := \max\{|Du(x)h|: |h| = 1\},$$

and

$$l(Du) := \min\{|Du(x)h|: |h| = 1\}.$$

See [7, p. 128] for the definition of quasiregular mappings in more general setting. A quasiregular homeomorphism is called quasiconformal.

Let $0 < \lambda_1^2 \leqslant \lambda_2^2 \leqslant \cdots \leqslant \lambda_n^2$ be the eigenvalues of the matrix $Du(x)Du(x)^t$. Here $Du(x)^t$ is the transpose of the matrix $Du(x)$. Then

$$J_u(x) = \prod_{k=1}^n \lambda_k, \tag{1.2}$$

$$|Du| = \lambda_n \tag{1.3}$$

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and

$$l(Du) = \lambda_1. \tag{1.4}$$

For the Hilbert–Schmidt norm of the matrix $Du(x)$, defined by

$$\|Du(x)\| = \sqrt{\operatorname{Trace}\big(Du(x)Du(x)^t\big)}$$

we have

$$\|Du(x)\| = \sqrt{\sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \bullet \frac{\partial u}{\partial x_k}} = \sqrt{\sum_{k=1}^{n} \left|\frac{\partial u}{\partial x_k}\right|^2} \tag{1.5}$$

and

$$\|Du(x)\| = \sqrt{\sum_{k=1}^{n} \lambda_k^2}. \tag{1.6}$$

Here $\bullet$ denotes the inner product between vectors. From (1.1), for a quasiregular mapping we have

$$\frac{\lambda_n}{\lambda_k}, \frac{\lambda_k}{\lambda_1} \leqslant K, \quad k = 1, \dots, n. \tag{1.7}$$

It is well known that if $u = (u_1, \dots, u_n)$ is a harmonic mapping defined in a region $\Omega$ of the Euclidean space $\mathbf{R}^n$, then $|u|^p$ is subharmonic for $p \geqslant 1$, and that, in the general case, is not subharmonic for $p < 1$. Let us prove this well-known fact. If $u$ is harmonic, then by a result in [4, Lemma 1.4] (see also [3, Eqs. (4.9)–(4.11)])

$$\Delta|u| = |u| \left\| D\left(\frac{u}{|u|}\right) \right\|^2.$$

So $\Delta|u| \geqslant 0$ for those points $x$, such that $u(x) \neq 0$. If $u(a) = 0$, then we consider the harmonic mapping $u_m(x) = u(x) + (1/m, 0, \dots, 0)$. Then $u_m(a) \neq 0$, and $\Delta|u_m(x)| \geqslant 0$ in some neighborhood of $a$. It follows from the definition of subharmonic functions that the uniform limit of a convergent sequence of subharmonic functions is still subharmonic. Since $|u_m(x)| \to |u(x)|$, it follows that $|u|$ is subharmonic in $a$. Since the function $g(s) = s^p$, is convex for $p \geqslant 1$, we obtain that $|u|^p$ is subharmonic providing that $u$ is harmonic. (For the above facts we refer to [2, Ch. 2].)

Recently, several authors have proved the following two propositions, which are the motivation for our study.

**Proposition 1.1.** *(See [5].) If $f$ is a $K$-quasiregular harmonic map in a plane domain, then $|f|^q$ is subharmonic for $q \geqslant 1 - K^{-2}$.*

**Proposition 1.2.** *(See [1].) If $f$ is a $K$-quasiregular harmonic map in a space domain, then $|f|^q$ is subharmonic for some $q = q(K, n) \in (0, 1)$.*

This paper is continuation of [1] in which Proposition 1.1 was extended to the $n$-dimensional setting. In [1] the authors prove only the existence of an exponent $q \in (0, 1)$ without giving the minimal value of $q$. Here we improve Proposition 1.2 by giving the optimal value of $q$. Our proof is completely different from those given in [1] and [5]. Moreover for the first time we consider the case $q < 0$.

Our proof is based on the following well-known explicit computation.

**Proposition 1.3.** *(See [6, Ch. VII 3, p. 217].) Let $u = (u_1, \dots, u_n) : \Omega \to \mathbf{R}^n$, be harmonic, let $\Omega_0 = \Omega \setminus u^{-1}(0)$, let $q \in \mathbf{R}$. Then for $x \in \Omega_0$*

$$\Delta|u|^q = q\left[|u|^{q-2} \sum_{k=1}^{n} |\nabla u_k|^2 + (q-2)|u|^{q-4} \sum_{k=1}^{n} \left(u \bullet \frac{\partial u}{\partial x_k}\right)\right].$$

**Proof.** Write $v := |u|^q = (u_1^2 + \dots + u_n^2)^p$, for $p := q/2$. A direct computation gives

$$\begin{aligned} v_{x_1} &= p\big(u_1^2 + \dots + u_n^2\big)^{p-1} \cdot (2u_1 u_{1x_1} + \dots + 2u_n u_{nx1}) \\ &= q\big(u_1^2 + \dots + u_n^2\big)^{p-1} \cdot (u_1 u_{1x_1} + \dots + u_n u_{nx1}), \end{aligned}$$

and further

$$v_{x_1x_1} = q\big\{2(p-1)\big(u_1^2+\cdots+u_n^2\big)^{p-2}\cdot(u_1u_{1x_1}+\cdots+u_{nx_1})^2 + \big(u_1^2+\cdots+u_n^2\big)^{p-1}\cdot\big[u_1u_{1x_1x_1}+(u_{1x_1})^2+\cdots+u_nu_{nx_1x_1}+(u_{nx_1})^2\big]\big\}.$$

Therefore

$$\begin{aligned}\Delta v &= v_{x_1x_1}+\cdots+v_{x_nx_n}\\ &= q\left\{|u|^{q-2}\left[(u_1\Delta u_1+\cdots+u_n\Delta u_n)+\left(\sum_{k=1}^n u_{1x_k}^2+\cdots+\sum_{k=1}^n u_{nx_k}^2\right)\right]+(q-2)|u|^{q-4}\sum_{k=1}^n\left(\sum_{j=1}^n u_ju_{jx_k}\right)^2\right\}\\ &= q\left\{|u|^{q-2}\left(\sum_{k=1}^n u_{1x_k}^2+\cdots+\sum_{k=1}^n u_{nx_k}^2\right)+(q-2)|u|^{q-4}\sum_{k=1}^n\left(\sum_{j=1}^n u_j\cdot\frac{\partial u_j}{\partial x_k}\right)^2\right\}\\ &= q|u|^{q-4}\left\{|u|^2\sum_{j=1}^n\left(\sum_{k=1}^n u_{jx_k}^2\right)+(q-2)\sum_{k=1}^n\left(\sum_{j=1}^n u_j\cdot\frac{\partial u_j}{\partial x_k}\right)^2\right\}\\ &= q|u|^{q-4}\left\{|u|^2\sum_{j=1}^n|\nabla u_j|^2+(q-2)\sum_{k=1}^n\left(u\bullet\frac{\partial u}{\partial x_k}\right)^2\right\}. \qquad\Box\end{aligned}$$

## 2. Main result

**Theorem 2.1.** *Let $u$ be $K$-quasiregular harmonic in $\Omega\subset\mathbb{R}^n$. Then the mapping $g(x)=|u(x)|^q$ is subharmonic in*

(1) *$\Omega$ for $q\geqslant\max\{1-\frac{n-1}{K^2},0\}$;*
(2) *$\Omega\setminus u^{-1}(0)$, for $q\leqslant 1-(n-1)K^2$.*

*Moreover for $1-(n-1)K^2<q<1-\frac{n-1}{K^2}$, there exists a $K$-quasiconformal harmonic mapping such that $|u|^q$ is not subharmonic.*

**Remark 2.2.** If $n=2$ then $1-\frac{n-1}{K^2}=1-K^{-2}$. Thus Theorem 2.1 is an extension of Proposition 1.1.

**Remark 2.3.** In the case $1\leqslant K\leqslant\sqrt{n-1}$ the function $|u|^q$ is subharmonic for all $q>0$.

**Proof of Theorem 2.1.** Let us fix such a map $u:\Omega\to\mathbf{R}^n$ and set $\Omega_0=\Omega\setminus u^{-1}\{0\}$. We have to find all positive real numbers $q$ such that $\Delta|u|^q\geqslant 0$ on $\Omega_0$. Since $u$ is quasiregular, the set $Z=\{x\in\Omega_0:\ \det Du(x)=0\}$ has measure zero (see [7]), it is also closed since $u$ is smooth. In particular, $\Omega_1=\Omega_0\setminus Z$ is dense in $\Omega_0$ and thus it suffices to prove that $\Delta|u|^q\geqslant 0$ on $\Omega_1$. From Proposition 1.3, we obtain

$$\Delta|u|^q = q\left[|u|^{q-2}\|Du\|^2+(q-2)|u|^{q-4}\left|\sum_{j=1}^n u_j\nabla u_j\right|^2\right]. \tag{2.1}$$

So we find all real $q$ such that

$$q\left(|u|^{q-2}\|Du\|^2+(q-2)|u|^{q-4}\left|\sum_{j=1}^n u_j\nabla u_j\right|^2\right)\geqslant 0.$$

If $q\geqslant 2$, then $\Delta|u|^q\geqslant 0$. Assume that $q\geqslant 0$ and $q<2$ such that

$$\left|\sum_{j=1}^n u_j(x)\nabla u_j(x)\right|^2\leqslant\frac{1}{2-q}\big|u(x)\big|^2\big\|Du(x)\big\|^2,\quad x\in\Omega_1.$$

After normalization, we see that it suffices to find all constants $q<2$ such that

$$\sup_{z\in S^{n-1}}\left|\sum_{j=1}^n z_j\nabla u_j(x)\right|^2\leqslant\frac{1}{2-q}\big\|Du(x)\big\|^2,\quad x\in\Omega_1. \tag{2.2}$$

Let $0 < \lambda_1^2 \leqslant \lambda_2^2 \leqslant \cdots \leqslant \lambda_n^2$ be the eigenvalues of the matrix $Du(x)Du(x)^t$. Then

$$\sup_{z\in S^{n-1}}\left|\sum_{j=1}^{n} z_j \nabla u_j(x)\right|^2 = \lambda_n^2, \tag{2.3}$$

$$\inf_{z\in S^{n-1}}\left|\sum_{j=1}^{n} z_j \nabla u_j(x)\right|^2 = \lambda_1^2 \tag{2.4}$$

and

$$\left\|Du(x)\right\|^2 = \sum_{k=1}^{n} \lambda_k^2. \tag{2.5}$$

Because $u$ is $K$-quasiregular from (1.7) we have

$$\frac{\lambda_n}{\lambda_k} \leqslant K, \quad k = 1, \dots, n-1. \tag{2.6}$$

Thus (2.2) can be written as

$$\lambda_n^2 \leqslant \frac{1}{2-q}\sum_{k=1}^{n} \lambda_k^2. \tag{2.7}$$

By (2.5) and (2.6) we get that, the inequality (2.7) is satisfied whenever

$$\frac{1}{1+\frac{n-1}{K^2}} \leqslant \frac{1}{2-q}, \tag{2.8}$$

i.e.

$$\max\left\{0, 1-\frac{n-1}{K^2}\right\} \leqslant q < 2. \tag{2.9}$$

If $q < 0$, then we should have

$$\inf_{z\in S^{n-1}}\left|\sum_{j=1}^{n} z_j \nabla u_j(x)\right|^2 \geqslant \frac{1}{2-q}\left\|Du(x)\right\|^2, \quad x \in \Omega_1, \tag{2.10}$$

i.e.

$$2-q \geqslant \sum_{k=1}^{n} \frac{\lambda_k^2}{\lambda_1^2}.$$

Because $u$ is $K$-quasiregular from (1.7)

$$\frac{\lambda_k}{\lambda_1} \leqslant K, \quad k = 2, \dots, n. \tag{2.11}$$

Thus if

$$q \leqslant 1-(n-1)K^2, \tag{2.12}$$

then (2.10) holds. To finish the proof we need the following lemma.

**Lemma 2.4.** *For any $1-(n-1)K^2 < q < 1-\frac{n-1}{K^2}$ there is a (linear) harmonic $K$-quasiconformal mapping $u$ such that $|u|^q$ is not subharmonic.*

**Proof.** Assume first that $q > 0$. We will consider linear mapping $u : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$u(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, Kx_n), \tag{2.13}$$

where $K \geqslant 1$. It is obviously harmonic and $K$-quasiconformal. If we put this mapping in formula (2.1) we get

$$\left[(n-1)+K^2\right]|u|^2 + (q-2)\left|\sum_{j=1}^{n-1} x_j e_j + K^2 e_n x_n\right|^2 \geqslant 0$$

which is equivalent to

$$\left(n-1+K^2\right)\left[\sum_{n=1}^{j-1} x_j^2 + K^2 x_n^2\right] + (q-2)\left|\sum_{j=1}^{n-1} x_j e_j + K^2 e_n x_n\right|^2 \geqslant 0.$$

By choosing $x_1 = \cdots = x_{n-1} = 0$ and $x_n = 1$, we obtain

$$\left(n-1+K^2\right)K^2 \geqslant (2-q)K^4$$

which is equivalent to

$$q \geqslant 1 - \frac{n-1}{K^2}.$$

For $q < 0$ we consider the linear mapping $u : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$u(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, x_n/K). \qquad \square \tag{2.14}$$

To finish the proof we only need to take $\tilde{u} = u|_{\Omega}$, where $u$ is defined in (2.13) respectively in (2.14). $\square$

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