Potential Anal (2012) 37:187–196
DOI 10.1007/s11118-011-9252-y

Quasi-Nearly Subharmonic Functions and Quasiconformal Mappings

Pekka Koskela · Vesna Manojlović

Received: 21 February 2011 / Accepted: 9 August 2011 / Published online: 27 August 2011

Abstract We prove that the composition of a quasi-nearly subharmonic function and a quasiregular mapping of bounded multiplicity is quasi-nearly subharmonic. Also, we prove that if $u \circ f$ is quasi-nearly subharmonic for all quasi-nearly subharmonic u and f satisfying some additional conditions, then f is quasiconformal. Similar results are further established for the class of regularly oscillating functions.

Keywords Quasi-nearly subharmonic functions · Quasiconformal mappings · Regularly oscillating functions

Mathematics Subject Classifications (2010) 31C05 · 30C65

1 Introduction and Results

Let Ω be a domain in the Euclidean space $\mathbf{R}^n$. If h is a function harmonic in Ω, then the function $|h|^p$, which need not be subharmonic in Ω for $0 < p < 1$, behaves like a subharmonic function: the inequality

$$|h(a)|^p \le \frac{C}{r^n} \int_{B(a,r)} |h|^p \, dm, \quad B(a,r) \subset \Omega,\ 0 < p < \infty, \tag{1.1}$$

Dedicated to Professor Miroslav Pavlović.

Pekka Koskela was supported by grants from the Academy of Finland and the second author by MN Project 174024, Serbia.

P. Koskela
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, 40014, Jyväskylä, Finland
e-mail: pkoskela@maths.jyu.fi

V. Manojlović (✉)
Faculty of Organizational Sciences, University of Belgrade, Jove Ilica 154, Belgrade, Serbia
e-mail: vesnam@fon.bg.ac.rs

holds, where $C \geq 1$, $B(a, r) = \{x : |x - a| < r\}$, and dm is the Lebesgue measure normalized so that $|B(0, 1)| := m(B(0, 1)) = 1$. The constant C in Eq. 1.1 depends only on n and p when $p < 1$, and $C = 1$ when $p \geq 1$. This fact is essentially due to Hardy and Littlewood (see [5, Theorem 5]), although they never formulated it. The proof was first given by Fefferman and Stein [4], and independently by Kuran [12]. It follows from Fefferman and Stein's proof that Eq. 1.1 remains true if $|h|$ is replaced by a nonnegative subharmonic function. Hence:

Theorem 1.1 *If $u \geq 0$ is a function subharmonic in a domain $\Omega \subset \mathbf{R}^n$, then*

$$u(a)^p \leq \frac{C}{r^n} \int_{B(a,r)} u^p \, dm, \qquad B(a, r) \subset \Omega, \; 0 < p < \infty, \tag{1.2}$$

where C depends only on p and n, when $p < 1$, and $C = 1$ when $p \geq 1$.

Let $u \geq 0$ be a locally bounded, measurable function on Ω. We say (see [13, 14]) that u is *C-quasi-nearly subharmonic* (abbreviated C-qns) if the following condition is satisfied:

$$u(a) \leq \frac{C}{r^n} \int_{B(a,r)} u \, dm, \qquad \text{whenever } B(a, r) \subset \Omega. \tag{1.3}$$

One can view Eq. 1.3 as a weak sub-mean value property. Besides nonnegative subharmonic functions it also holds for nonnegative subsolutions to a large family of second order elliptic equations, see [6]. In fact, Eq. 1.3 is typically proven as a step towards Harnack inequalities for second order elliptic equations, using the Moser iteration scheme. Notice that u is quasi-nearly subharmonic if and only if u is everywhere dominated by its centered minimal function [2].

Our first result is an invariance property.

Theorem 1.2 *If $u \geq 0$ is a C-quasi-nearly subharmonic function defined on a domain $\Omega' \subset \mathbf{R}^n$, $n \geq 2$, and f is a K-quasiregular mapping, with bounded multiplicity N, from a domain Ω onto Ω', then the function $u \circ f$ is C_1-quasi-nearly subharmonic in Ω, where C_1 only depends on K, C, N, and n.*

Remark 1.1 The hypothesis of bounded multiplicity of f is necessary as the following example shows. Let $f(z) = e^z$, $\Omega = \mathbb{C}$, $\Omega' = \mathbb{C} \setminus \{0\}$,

$$E = \bigcup_{j \geq 2} [\exp(2^j), \exp(2^j + 1)],$$

and $u(w) = \chi_E(|w|)$. Then it is easy to check that u is quasi-nearly subharmonic in Ω' but $u \circ f$ is not quasi-nearly subharmonic in Ω.

Above quasiregularity requires that f is continuous, the component functions of f belong locally to the Sobolev class $W^{1,n}$ and that there is a constant $K \geq 1$ so that

$$|Df(x)|^n \leq K J(x, f)$$

holds almost everywhere in Ω. Injective quasiregular mappings are called quasiconformal. It was previously only known that the invariance holds under conformal mappings in the planar case [10] and under bi-Lipschitz mappings [3] in all dimensions. Let us consider the above morphism property in more detail.

Definition 1.1 Let Ω and Ω' be subdomains of $\mathbf{R}^n$. A mapping $f : \Omega \mapsto \Omega'$ is a *quasi-nearly subharmonic-morphism* if there is a constant $C < \infty$ such that for every quasi-nearly subharmonic u defined in Ω' we have

$$\|u \circ f\|_{\mathrm{qns}} \leq C\|u\|_{\mathrm{qns}},$$

where

$$\|u\|_{\mathrm{qns}} = \inf\left\{C \geq 0 : u(a) \leq \frac{C}{r^n}\int_{B(a,r)} u\,dm \text{ for all } a \in \Omega',\ 0 < r \leq d(x, \partial\Omega')\right\}.$$

If the above holds with a constant C, we call f a C-quasi-nearly subharmonic-morphism. Finally, f is a *strong qns-morphism* if there is a constant C so that f restricted to any domain $G \subset \Omega$, $f : G \mapsto G'$, is a C-qns-morphism.

Theorem 1.3 *Let Ω, $\Omega' \subset \mathbb{R}^n$, $n \geq 2$, be domains. Then a homeomorphism $f : \Omega \mapsto \Omega'$ is a strong qns-morphism if and only if f is quasiconformal.*

If we assume sufficient a priori regularity for f, a version of Theorem 1.3 holds also for qns-morphisms. The reader may wish to compare this with related quasiconformal invariance properties for other function classes [1, 15, 17–19].

Theorem 1.4 *Let $n \geq 2$ and let $f : \Omega \mapsto \Omega'$ be a quasi-nearly subharmonic-morphism that belongs to $W^{1,n}_{\mathrm{loc}}$. If, additionally, $J(x, f) \geq 0$ almost everywhere, then f is quasiregular.*

Each quasiregular mapping f is either constant or both open and discrete; in the latter case the multiplicity of f is locally finite. The condition $J(x, f) \geq 0$ in Theorem 1.4 cannot be dropped, as the mapping $f(x, y) = (x, |y|)$ is a planar (strong) quasi-nearly subharmonic-morphism. The Sobolev regularity assumption can be slightly relaxed: if f above is a C-quasi-nearly subharmonic-morphism, then local p-integrability of the distributional derivatives suffices for $p = p(n, C) < n$; this can be inferred from the proof of Theorem 1.4 using [9]. In the planar, injective setting, even $W^{1,1}_{\mathrm{loc}}$ suffices.

Let us close this introduction by commenting on the invariance of a related function class, introduced in [13].

Definition 1.2 A function $u : \Omega' \mapsto \mathbb{R}^k$ is said to be *regularly oscillating* if

$$\operatorname{Lip} u(x) \leq Cr^{-1} \sup_{y \in B(x,r) \subset \Omega'} |u(y) - u(x)|, \quad x \in \Omega',\ B(x, r) \subset \Omega', \tag{1.4}$$

where $C \geq 0$ is a constant independent of x and r. Here

$$\operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|}.$$

(This notation is borrowed from [11].) Note that $\operatorname{Lip} u(x) = |\operatorname{grad} u(x)|$ if u is differentiable at x. The smallest C satisfying Eq. 1.4 will be denoted by $\|u\|_{\mathrm{ro}}$.

We have the following invariance.

Theorem 1.5 *Let $f : \Omega \mapsto \Omega'$ be quasiregular, regularly oscillating and of bounded multiplicity in Ω. If u is regularly oscillating in Ω', then $u \circ f$ is regularly oscillating in Ω with $\|u \circ f\|_{\mathrm{ro}} \leq C'\|u\|_{\mathrm{ro}}$, where C' depends only on the multiplicity of f.*

The assumption that f be regularly oscillating is necessary, as seen by noticing that the coordinate projections are regularly oscillating; not all quasiregular mappings are regularly oscillating. In the case of an analytic function, this can naturally be dropped. Similarly to Theorem 1.4, quasiregularity is necessary if we assume that $J(x, f) \geq 0$ almost everywhere, but no a priori Sobolev regularity is needed because regularly oscillating functions and mappings are locally Lipschitz continuous. The invariance property of Theorem 1.5 was established in [10] when f is conformal (and $n = 2$).

Remark 1.2 The assumption of bounded multiplicity of f in Theorem 1.5 is necessary as in the case of Theorem 1.2. To see this simply let $f(z) = e^z$, $\Omega = \mathbb{C}$, $\Omega' = \mathbb{C} \setminus \{0\}$,

$$E = \bigcup_{j \geq 2} [\exp(2^j), \exp(2^j + 1)],$$

and $v(w) = \int_0^{|w|} \chi_E(t)\, dt$. Then v is regularly oscillating but $v \circ f$ is not.

2 Proof of the Theorem 1.2

For the proof we need some lemmas. The first says that if u^p is qns for some p, then so is u.

Lemma 2.1 [13] *If u is C-quasi-nearly subharmonic, and $p > 0$, then u^p is $C_1(C, n)$-quasi-nearly subharmonic.*

We also need the following lemma that can be distilled from the arguments in [7]. For the sake of completeness, we give a short proof below.

Lemma 2.2 *Let $f : \Omega \mapsto \Omega'$ be K-quasiregular and of bounded multiplicity N. Let $x \in \Omega$ and $0 < r \leq \frac{1}{2} d(x, \partial\Omega r))$. Then*

$$d\big(f(x), \partial f(B(x,r))\big) \geq \delta \sup_{y \in \overline{B}(x,r)} |f(y) - f(x)|,$$

where $\delta = \delta(n, K, N)$.

Proof Let $x \in \Omega$ and let $0 < r \leq \frac{1}{2}d(x, \partial\Omega)$. Now $f(x)$ is an interior point of $f(B(x,r))$ because f is open. Moreover

$$\sup_{y \in \overline{B}(x,r)} |f(y) - f(x)| = |f(z) - f(x)|$$

for some $z \in \partial B(x,r)$, and

$$0 < d(f(x), \partial f(B(x,r)) = |f(\omega) - f(x)|$$

for some $\omega \in \partial B(x,r)$.

Let $E = [f(x), f(\omega)]$ be the segment between $f(w)$ and $f(\omega)$, and F be a segment that joins $f(z)$ to $\partial\Omega'$ (or to infinity) outside the ball

$$B(f(x), |f(z) - f(\omega)|).$$

We may assume that

$$|f(z) - f(x)| \geq 2|f(\omega) - f(x)|.$$

Let

$$u(y) = \begin{cases} 1, & y \in \overline{B}(f(x), |f(\omega) - f(x)|), \\ 0, & y \in B^c(f(x), |f(z) - f(x)|), \\ \log \frac{|f(z)-f(x)|}{|y-f(x)|} \Big/ \log \frac{|f(z)-f(x)|}{|f(\omega)-f(x)|} & \text{elsewhere.} \end{cases}$$

Then, by a change of variables, see page 21 in [16],

$$\begin{aligned} \int_\Omega |\nabla(u \circ f)|^n \, dm &\leq K \int_{\Omega'} |(\nabla u)(f(y))|^n J_f(y) \, dm(y) \\ &\leq KN \int_{\Omega'} |\nabla u|^n \, dm \\ &\leq \frac{KNC_n}{\log^{n-1} \dfrac{|f(z) - f(x)|}{|f(\omega) - f(x)|}}. \end{aligned}$$

On the other hand, since f is open, the set $f^{-1}([f(x), f(\omega)])$ contains a continuum joining x to $\partial B(x,r)$, and $f^{-1}(F)$ a continuum joining $\partial B(x,r)$ to $\partial B(x, \frac{3}{2}r)$. By usual capacity estimates, see e.g. [6]

$$\int_\Omega |\nabla(u \circ f)|^n \, dm \geq \delta_0(n, K) > 0.$$

The claim follows. □

As an immediate consequence of Lemma 2.2 we have:

Lemma 2.3 *Let $B = B(0,1)$ and let $f : 2B \mapsto \Omega'$ be a K-qr mapping with bounded multiplicity N and such that $f(0) = 0$. Then there exist $\rho \in (0,1)$ and $R > 0$ such that $B(0, R) \supset f(B) \supset B(0, \rho)$, where $R/\rho \leq 1/\delta$, and δ depends only on K, n, N.*

Finally we need the following fundamental fact:

Lemma 2.4 [8, p. 258] *Under the hypotheses of Lemma 2.3, there exists* $p > 1$ *such that*

$$\Big(\int_B J(y,\, f)^p\, dm\Big)^{1/p} \le C \int_B J(y,\, f)\, dm,$$

where p *depends only on* K, N *and* n.

Proof of Theorem 1.2 As is easily seen, the proof reduces to the case $\Omega = B(0, 2)$, $f(0) = 0$. Let $B = B(0, 1)$ and write $v = u \circ f$. By using translations and rotations, we see that it is enough to prove that

$$v(0) \le C \int_B v(y)\, dm(y),$$

where $C = C(K, n, \|u\|_{\text{qns}})$. By Lemma 2.1, it suffices to find $q = q(K, N, n) \ge 1$ so that

$$v(0) \le C\Big(\int_B v(y)^q\, dm(y)\Big)^{1/q}. \tag{2.1}$$

To prove this, we start from Hölder's inequality:

$$\int_B v(y) J(y,\, f)\, dm(y) \le \Big(\int_B v(y)^q\, dm(y)\Big)^{1/q} \Big(\int_B J(y,\, f)^p\, dm(y)\Big)^{1/p},$$

where $p = q/(q-1)$. By a change of variables, see page 21 in [16], we have

$$\begin{aligned}\int_B v(y) J(y,\, f)\, dm(y) &= \int_{f(B)} u(y)\, N(y,\, f,\, B)\, dm(y)\\ &\ge \int_{f(B)} u(y)\, dm(y)\\ &\ge \int_{B(0,\rho)} u(y)\, dm(y)\\ &\ge c\rho^n u(0) = c\rho^n v(0).\end{aligned}$$

Here we have used Lemma 2.3 and the hypothesis that u is qns. On the other hand, by Lemmas 2.4 and 2.3, we have

$$\begin{aligned}\Big(\int_B J(y,\, f)^p\, dm(y)\Big)^{1/p} dy &\le C \int_B J(y,\, f)\, dm(y)\\ &= C \int_{f(B)} N(y,\, f,\, B)\, dm(y)\\ &\le CN|\, f(B)|\\ &\le CN|B(0,\, R)| = CNk_n R^n.\end{aligned}$$

Combining these inequalities, we obtain

$$c\rho^n v(0) \le CNk_n R^n \Big(\int_B v(y)^q\, dm(y)\Big)^{1/q}.$$

Hence

$$v(0) \le \frac{CNk_n R^n}{c\rho^n}\Big(\int_B v(y)^q\, dm(y)\Big)^{1/q}.$$

Now the desired result follows from the inequality $R/\rho \le 1/\delta$, where δ depends only on K, n, N. □

3 Proof of Theorem 1.4

Even though our definition of quasiregular mappings requires them to be continuous, this condition is superfluous and it suffices to show that there exists $K \ge 1$ so that

$$|Df(x)|^n \le KJ(x, f)$$

holds almost everywhere, see e.g. page 177 in [16]. Next, every mapping f with Sobolev regularity $W^{1,1}_{\mathrm{loc}}$ is approximatively differentiable almost everywhere. That is, for almost every x_0 and every $\epsilon > 0$, the set

$$A_\epsilon := \left\{x : \frac{|f(x) - f(x_0) - Df(x_0)(x - x_0)|}{|x - x_0|} < \epsilon\right\}$$

has density one at x_0, see e.g. page 140 in [20]. Because of our a priori Sobolev regularity, it thus suffices to show the above distortion inequality at every such point x_0.

For simplicity, we only give the proof in the planar case, assuming differentiability instead of approximate differenentiability. The higher dimensional setting and the switch to approximate differentiability only require technical modifications that should be obvious to the reader after examining the argument below. Thus suppose that f is differentiable at x_0.

Case (a) Suppose f is differentiable at x_0 with $J_f(x_0) \ne 0$. In some coordinate systems we have

$$Df(x_0) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Assume $0 < a < b$. We want to show that b/a is bounded. Consider the function

$$u(\omega) = \chi_{\{\omega = x' + iy' : 0 \le y' \le x'\}}(\omega - f(x_0)).$$

Then $\|u\|_{\mathrm{qns}} = 8$. Now

$$\begin{aligned} T^{-1}\big(\{\omega = x' + iy' : 0 \le y' \le x'\}\big) &= \{z = x + iy : 0 \le by \le ax\} \\ &= \{z = x + iy : 0 \le y \le (a/b)x\} \end{aligned}$$

for the linear transformation T associated to $Df(x_0)$. Now $u \circ f(x_0) = 1$. If $r > 0$ is such that $B(x_0, r) \subset \Omega$, we conclude from the morphism property of f that

$$\begin{aligned} 1 &\le \frac{C}{r^2} \int_{B(0,r)} u \circ f \, dm \\ &= \frac{C}{r^2} \frac{1}{2} r^2 \arctan \frac{a}{b} + o(r) \\ &\to \frac{C}{2} \frac{a}{b}, \end{aligned}$$

when $r \to 0$, where $C > 0$ comes from the morphism property. Hence $b/a \le C/2$.

Case (b) Now suppose that f is differentiable at x_0 with $J_f(x_0) = 0$. We want to prove that $Df(x_0) = 0$. We argue by contradiction: suppose $Df(x_0) \ne 0$. We may assume

$$Df(x_0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define

$$u(\omega) = \chi_{\{\omega = x' + iy' : 0 \le |y'| \le x'\}}(\omega - f(x_0)).$$

Then $\|u\|_{\mathrm{qns}} = 4$, and

$$f^{-1}(t + it + f(x_0)) = s_1(t) + is_2(t) + x_0.$$

where $\lim_{t \to 0} \dfrac{s_2(t)}{s_1(t)} = 0$. But then there is no $C > 0$ such that

$$(u \circ f)(x_0) \le \frac{C}{r^2} \int_{B(x_0,r)} u \circ f \, dm$$

for all small $r > 0$, which contradicts the morphism property of f.

4 Proof of Theorem 1.3

It is an immediate consequence of Theorem 1.2 that quasiconformality of the homeomorphism f is a sufficient condition for f to be a strong qns-morphism.

In the other direction, it suffices to prove that $f^{-1} : \Omega' \mapsto \Omega$ is quasiconformal. Thus it suffices to verify the existence of $H < \infty$ such that

$$\limsup_{r \to 0} \frac{\operatorname{diam}(f^{-1}(\overline{B}(y, r)))^n}{|f^{-1}(\overline{B}(y, r))|} \le H \tag{4.1}$$

for all $y \in \Omega'$, see page 64 in [8].

To simplify our notation, we write $x' = f(x)$ for $x \in \Omega$, in what follows.

Fix $y' \in \Omega'$ and let $r > 0$. Towards proving Eq. 4.1, we may assume that r is so small that

$$B(y, 2 \operatorname{diam}(f^{-1}(\overline{B}(y', 2r)))) \subset \Omega.$$

Fix $y'_0 \in \partial B(y', 2r)$ and pick $y'_1 \in \partial B(y', r)$ so that

$$|y'_0 - y'_1| = \max_{\omega' \in \partial B(y', r)} |\omega' - y'_0|.$$

Set $G' = \Omega' \setminus \{y'_0\}$ and $G = \Omega \setminus \{y_0\}$. Now $B(y_1, |y_1 - y_0|/2) \subset G$ and

$$\operatorname{diam}\,(f^{-1}(\overline{B}(y', r)) \leq 2|y_1 - y_0|. \tag{4.2}$$

Define $u(\omega') = \chi_{\overline{B}(y',r)}(\omega')$ for $\omega' \in G'$. Then u is qns in G' with $\|u\|_{\text{qns}} \leq 3^n$. Since f is C-qns-morphism in G, we conclude that

$$u \circ f(y_1) \leq C3^n \frac{1}{|B(y_1, |y_1 - y_0|/2)|} \int_{B(y_1, |y_1 - y_0|/2)|} u \circ f\, dm$$

$$= C3^n \frac{|f^{-1}(\overline{B}(y', r)) \cap B(y_1, |y_1 - y_0|/2)|}{|B(y_1, |y_1 - y_0|/2)|}$$

Recalling Eq. 4.2 and that $u \circ f(y_1) = u(y'_1) = 1$, we arrive at

$$\operatorname{diam}\,(f^{-1}(\overline{B}(y', r))^n \leq CC_n |f^{-1}(\overline{B}(y', r))|$$

as desired.

5 Proof of Theorem 1.5

5.1 Proof of Theorem 1.5

Let $x \in \Omega$ and $0 < r < \frac{1}{2} d(x, \partial\Omega)$. Since the mapping f is regularly oscillating and quasiregular we have, by Lemma 2.2,

$$\operatorname{Lip} f(x) \leq Cr^{-1} \sup_{y \in \overline{B}(x,r)} |f(y) - f(x)| \leq \frac{C}{\delta} r^{-1} d(f(x), \partial f(B(x, r))).$$

Recall that non-constant quasiregular mappings are open. Since u is regularly oscillating and $d(f(x), \partial f(B(x, r))) > 0$, we have that

$$\operatorname{Lip} u(f(x)) \leq \hat{C} d(f(x), \partial f(B(x, r)))^{-1} \sup_{z \in B(f(x), d(f(x), \partial f(B(x,r)))} |u(f(x)) - z|$$

$$\leq \hat{C} d(f(x), \partial f(B(x, r)))^{-1} \sup_{y \in B(x,r)} |u \circ f(y) - u \circ f(x)|.$$

Now we have

$$\begin{aligned}\operatorname{Lip}\,(u \circ f)(x) &\leq \operatorname{Lip}\,(u(f(x)) \operatorname{Lip} f(x)\\ &\leq \hat{C} d(f(x), \partial f(B(x, r)))^{-1} \sup_{y \in B(x,r)} |u \circ f(y) - u \circ f(x)|\\ &\quad \times \frac{C}{\delta} r^{-1} d(f(x), \partial f(B(x, r)))\\ &= C' r^{-1} \sup_{y \in B(x,r)} |u \circ f(y) - u \circ f(x)|.\end{aligned}$$

This completes the proof of the theorem. □

References

1. Astala, K.: A remark on quasiconformal mappings and BMO-functions. Mich. Math. J. **30**, 209–212 (1983)
2. Cruz-Uribe, D.: The minimal operator and the geometric maximal operator in $\mathbf{R}^n$. Stud. Math. **144**, 1–37 (2001)
3. Dovgoshey, O., Riihentaus, J.: Bi-Lipschitz mappings and quasinearly subharmonic functions. Int. J. Math. Math. Sci. **2010**, 1–8 (2010)
4. Fefferman, C., Stein, E.M.: H^p-spaces of several variables. Acta Math. **129**, 137–193 (1972)
5. Hardy, G.H., Littlewood, J.E.: Some properties of conjugate functions. J. Reine Angew. Math. **167**, 405–423 (1931)
6. Heinonen, J., Kilpeläinen, T., Martio, O.: Nonlinear potential theory of degenerate elliptic equations. Oxford mathematical monographs. Oxford Science Publications. The Clarendon Press, pp. vi+363. Oxford University Press, New York (1993)
7. Heinonen, J., Koskela, P.: Weighted Sobolev and Poincaré inequalities and quasiregular mappings of polynomial type. Math. Scand. **77**, 251–271 (1995)
8. Heinonen, J., Koskela, P.: Definitions of quasiconformality. Invent. Math. **120**, 61–79 (1995)
9. Iwaniec, T.: p-harmonic tensors and quasiregular mappings. Ann. Math. **136**(2), 589–624 (1992)
10. Kojić, V.: Quasi-nearly subharmonic functions and conformal mappings. Filomat (Niš) **21**(2), 243–249 (2007)
11. Koskela, P.: Lectures on quasiconformal and quasisymmetric mappings, p. 100 (2009)
12. Kuran, Ü.: Subharmonic behaviour of $|h|^p$ ($p > 0, h$ harmonic). J. Lond. Math. Soc. **8**(2), 529–538 (1974)
13. Pavlović, M.: On subharmonic behaviour and oscillation of functions on balls in $\mathbf{R}^n$. Publ. Inst. Math. (Belgrade) **55**, 18–22 (1994)
14. Pavlović, M., Riihentaus, J.: Classes of quasi-nearly subharmonic functions. Potential Anal. **29**(1), 89–104 (2008)
15. Reimann, H.M.: Functions of bounded mean oscillation and quasiconformal mappings. Comment. Math. Helv. **49**, 260–276 (1974)
16. Rickman, S.: Quasiregular mappings. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Results in Mathematics and Related Areas (3), vol. 26, pp. x+213. Springer, Berlin (1993)
17. Staples, S.G.: Maximal functions, A_∞ measures and quasiconformal maps. Proc. Am. Math. Soc. **113**(3), 689–700 (1991)
18. Staples, S.G.: Doubling measures and quasiconformal maps. Comment. Math. Helv. **67**, 119–128 (1992)
19. Uchiyama, A.: Weight functions of the class (A_∞) and quasi-conformal mappings. Proc. Jpn. Acad. **51**, 811–814 (1975)
20. Ziemer, W.P.: Weakly differentiable functions. Sobolev spaces and functions of bounded variation. Graduate Texts in Mathematics, vol. 120, pp. xvi+308. Springer-Verlag, New York (1989)