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On order-Lipschitz mappings in Banach spaces without normalities of involving cones

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Abstract

We prove a new fixed point theorem of order-Lipschitz mappings in Banach spaces without assumption of normalities of the involving cones, which presents a positive answer to a problem raised in [S. Jiang, Z. Li, Fixed Point Theory Appl., **2016** (2016), 10 pages] and improves the corresponding results of Krasnoselskii and Zabreiko's and Zhang and Sun's since the normality of the involving cone is removed. ©2017 All rights reserved.

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1. Introduction and preliminaries

Let P be a cone of a Banach space $(E, \|\cdot\|)$, $D \subset E$ and $\preceq$ the partial order in E deduced by P. Recall that a mapping $T : D \to E$ is an order-Lipschitz mapping, if there exist two linear bounded mappings $A, B : P \to P$ such that

$$-B(x-y) \preceq Tx - Ty \preceq A(x-y), \quad \forall\, x, y \in D, \quad y \preceq x. \tag{1.1}$$

In particular, when $A = B$, Krasnoselskii and Zabreiko [4] proved the following fixed point theorem of order-Lipschitz mappings by using the Banach contraction principle.

Theorem 1.1 ([4]). *Let P be a normal solid cone of a Banach space $(E, \|\cdot\|)$ and $T : E \to E$ an order-Lipschitz mapping such that (1.1) is satisfied with linear bounded mappings A and B. If $A = B$ and $\|A\| < 1$, then T has a unique fixed point $x^* \in E$, and $x_n \xrightarrow{w} x^*$ for each $x_0 \in E$, where $\{x_n\} = O(T, x_0)$ and $O(T, x_0)$ denotes the Picard iterative sequence of T at x_0, i.e., $x_n = T^n x_0$ for each n.*

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Afterward, Zhang and Sun [7] showed Theorem 1.1 is still valid in the case that the spectral radius $r(A) < 1$, and obtained the following fixed point result.

Theorem 1.2 ([7]). *Let P be a normal solid cone of a Banach space $(E, \|\cdot\|)$ and $T : E \to E$ an order-Lipschitz mapping such that (1.1) is satisfied with linear bounded mappings A and B. If $A = B$ and $r(A) < 1$, then T has a unique fixed point $x^* \in E$, and $x_n \overset{w}{\to} x^*$ for each $x_0 \in E$, where $\{x_n\} = O(T, x_0)$ and $O(T, x_0)$ denotes the Picard iterative sequence of T at x_0, i.e., $x_n = T^n x_0$ for each n.*

In particular when A, B are nonnegative real numbers, Sun [6] proved the following fixed point theorem by using the sandwich theorem in the sense of norm-convergence.

Theorem 1.3 ([6]). *Let P be a normal cone of a Banach space $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : [u_0, v_0] \to E$ an order-Lipschitz mapping such that*

$$u_0 \preceq Tu_0, \qquad Tv_0 \preceq v_0, \tag{1.2}$$

and (1.1) is satisfied with nonnegative real numbers A and B. If $A \in [0, 1)$ and $B \in [0, +\infty)$, then T has a unique fixed point $x^* \in [u_0, v_0]$, and $x_n \overset{\|\cdot\|}{\to} x^*$ for each $x_0 \in [u_0, v_0]$, where $\{x_n\} = O(T, x_0)$.

Note that the normality of P in Theorems 1.1 and 1.2 is essential for the completeness of $(E, \|\cdot\|_0)$, where $\|\cdot\|_0$ is a new norm in E defined by $\|x\|_0 = \inf_{u \in P}\{\|u\| : -u \preceq x \preceq u\}$, which leads to that the Banach contraction principle is applicable there. And the normality of P in Theorem 1.3 is essential for ensuring that the sandwich theorem holds in the sense of norm-convergence, which makes an important role in its proof. It is well-known that if P is non-normal then the sandwich theorem does not hold in the sense of norm-convergence, and consequently, the method used in [6] becomes invalid.

In most of the existing works concerned with fixed point theory of order-Lipschitz mappings, the cone is necessarily assumed to be normal. Recently, Jiang and Li [3] considered fixed point theory of order-Lipschitz mappings without assuming the normality of P. By introducing the concept of Picard-completeness and using the sandwich theorem in the sense of w-convergence, they proved the following fixed point theorem of order-Lipschitz mappings in Banach algebras.

Theorem 1.4 ([3]). *Let P be a solid cone of a Banach algebra $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$, and $T : [u_0, v_0] \to E$ an order-Lipschitz mapping such that (1.1) and (1.2) are satisfied with $A, B \in P$. If $r(A) < 1$ and $B = \theta$, then T has a unique fixed point $x^* \in [u_0, v_0]$, and $x_n \overset{w}{\to} x^*$ for each $x_0 \in [u_0, v_0]$, where $\{x_n\} = O(T, x_0)$.*

In [3] , the authors failed to improve Theorem 1.1 to the case that the cone is non-normal. Instead, they raised a problem whether the normality of P in Theorem 1.1 could be removed. In the paper, we present a positive answer to this problem, and prove that Theorems 1.1 and 1.2 are still valid without assuming the normality of P. In addition, we give an suitable example to show the usability of our theorem.

Let $(E, \|\cdot\|)$ be a Banach space. A nonempty closed subset P of E is a cone, if it is such that $ax + by \in P$ for each $x, y \in P$ and each $a, b \geqslant 0$, and $P \cap (-P) = \{\theta\}$, where θ is the zero element of E. Each cone P of a Banach space E determines a partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$ for each $x, y \in X$. For each $u_0, v_0 \in E$ with $u_0 \preceq v_0$, we set $[u_0, v_0] = \{u \in E : u_0 \preceq u \preceq v_0\}$, $[u_0, +\infty) = \{x \in E : u_0 \preceq x\}$ and $(-\infty, v_0] = \{x \in E : x \preceq v_0\}$. A cone P is solid [1] if $\mathrm{int}P$ is nonempty, where $\mathrm{int}P$ denotes the interior of P. For each $x, y \in E$ with $y - x \in \mathrm{int}P$, we write $x \ll y$.

A cone P is normal [1], if there is some positive number N such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leqslant N\|y\|$, and the minimal N is called a normal constant of P. Note that an equivalent condition of a normal cone is that $\inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0$, then a cone P is non-normal, if and only if there exist $\{u_n\}, \{v_n\} \subset P$ such that $u_n + v_n \overset{\|\cdot\|}{\to} \theta \not\Rightarrow u_n \overset{\|\cdot\|}{\to} \theta$. This yields that the sandwich theorem does not hold in the sense of norm-convergence.

Definition 1.5 ([3]). Let P be a solid cone of a Banach space $(E, \|\cdot\|)$, $\{x_n\} \subset E$ and $D \subset E$.

(i) The sequence $\{x_n\}$ is w-convergent, if for each $\epsilon \in \mathrm{int}P$, there exist some positive integer n_0 and $x \in E$ such that $x - \epsilon \ll x_n \ll x + \epsilon$ for each $n \geqslant n_0$ (denote $x_n \overset{w}{\to} x$ and x is called a w-limit of $\{x_n\}$);

(ii) the sequence $\{x_n\}$ is w-Cauchy, if for each $\epsilon \in \text{int}P$, there exists some positive integer n_0 such that $-\epsilon \ll x_n - x_m \ll \epsilon$ for each $m, n \geqslant n_0$, i.e., $x_n - x_m \overset{w}{\to} \theta (m, n \to \infty)$;

(iii) the subset D is w-closed, if for each $\{x_n\} \subset D$, $x_n \overset{w}{\to} x$ implies $x \in D$.

The following lemmas are very important for our further discussions.

Lemma 1.6 ([3]). *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $\{x_n\}$ a w-convergent sequence of E. Then $\{x_n\}$ has a unique w-limit.*

Lemma 1.7 ([5, 2]). *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $\{x_n\}, \{y_n\}, \{z_n\} \subset E$ with $x_n \preceq y_n \preceq z_n$ for each n. If $x_n \overset{w}{\to} z$ and $z_n \overset{w}{\to} z$, then $y_n \overset{w}{\to} z$.*

Lemma 1.8 ([5, 2]). *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $x_n \subset E$. Then $x_n \overset{\|\cdot\|}{\to} x$ implies $x_n \overset{w}{\to} x$. Moreover, if P is normal then $x_n \overset{w}{\to} x \Leftrightarrow x_n \overset{\|\cdot\|}{\to} x$.*

Lemma 1.9 ([1]). *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$). Then there is $\tau > 0$ such that for each $x \in E$, there exist $y, z \in P$ with $\|y\| \leqslant \tau\|x\|$ and $\|z\| \leqslant \tau\|x\|$ such that $x = y - z$.*

Definition 1.10 ([3]). Let P be a solid cone of a Banach space $(E, \|\cdot\|)$, $x_0 \in E$ and $T : E \to E$. If the Picard iterative sequence $O(T, x_0)$ is w-convergent provided that it is w-Cauchy, then T is said to be Picard-complete at x_0. If T is Picard-complete at each $x \in E$, then it is said to be Picard-complete on E.

Remark 1.11.

(i) If $O(T, x_0)$ is w-convergent, then T is certainly Picard-complete at x_0.

(ii) If P is a normal cone then each mapping $T : E \to E$ is Picard-complete on E by Lemma 1.8.

2. Main results

Theorem 2.1. *Let P be a solid cone of a Banach space $(E, \|\cdot\|)$ and $T : E \to E$ an order-Lipschitz mapping such that (1.1) is satisfied with linear bounded mappings A and B. If $A = B$, $r(A) < 1$ and*

$$E_{T-C} = \{x \in E : T \textit{ is Picard-complete at } x\} \neq \emptyset,$$

then T has a unique fixed point $x^* \in E$. Moreover, for each $x_0 \in E_{T-C}$, we have $x_n \overset{w}{\to} x^*$, where $\{x_n\} = O(T, x_0)$.

Proof.
Step 1. We show that for each $x, y \in X$, there exists $u \in P$ such that

$$-u \preceq x - y \preceq u, \tag{2.1}$$

and

$$-A^n u \preceq T^n x - T^n y \preceq A^n u, \quad \forall\, n. \tag{2.2}$$

It follows from the solidness of P and Lemma 1.9 that there is a $\tau > 0$ such that for each $x \in E$, there exist $y, z \in P$ with $\|y\| \leqslant \tau\|x\|$ and $\|z\| \leqslant \tau\|x\|$ such that $x = y - z$, and so we have

$$-(y + z) \preceq x \preceq y + z.$$

This shows that for each $x \in E$, there exists $u \in P$ such that

$$-u \preceq x \preceq u,$$

and so for each $x, y \in E$, there exists $u \in P$ such that (2.1) is satisfied. For each $x, y \in E$, by (2.1) we get

$$\frac{x+y-u}{2} \preceq x, \qquad \frac{x+y-u}{2} \preceq y.$$

Thus by (1.1), we have

$$-A(\frac{x-y+u}{2}) \preceq Tx - T(\frac{x+y-u}{2}) \preceq A(\frac{x-y+u}{2}), \tag{2.3}$$

and

$$-A(\frac{y-x+u}{2}) \preceq Ty - T(\frac{x+y-u}{2}) \preceq A(\frac{y-x+u}{2}),$$

which can be rewritten as

$$-A(\frac{y-x+u}{2}) \preceq T(\frac{x+y-u}{2}) - Ty \preceq A(\frac{y-x+u}{2}). \tag{2.4}$$

By adding (2.3) and (2.4), we get

$$-Au \preceq Tx - Ty \preceq Au,$$

i.e., (2.2) holds for $n = 1$. Suppose that (2.2) holds for n, then

$$\frac{T^n x + T^n y - A^n u}{2} \preceq T^n x, \qquad \frac{T^n x + T^n y - A^n u}{2} \preceq T^n y.$$

Moreover by (1.1), we have

$$-A(\frac{T^n x - T^n y + A^n u}{2}) \preceq T^{n+1}x - T(\frac{T^n x + T^n y - A^n u}{2}) \preceq A(\frac{T^n x - T^n y + A^n u}{2}), \tag{2.5}$$

and

$$-A(\frac{T^n y - T^n x + A^n u}{2}) \preceq T^{n+1}y - T(\frac{T^n x + T^n y - A^n u}{2}) \preceq A(\frac{T^n y - T^n x + A^n u}{2}),$$

which can be rewritten as

$$-A(\frac{T^n y - T^n x + A^n u}{2}) \preceq T(\frac{T^n x + T^n y - A^n u}{2}) - T^{n+1}y \preceq A(\frac{T^n y - T^n x + A^n u}{2}). \tag{2.6}$$

By adding (2.5) and (2.6), we get $-A^{n+1}u \preceq T^{n+1}x - T^{n+1}y \preceq A^{n+1}u$ for each $x, y \in E$, i.e., (2.2) holds for $n+1$. Thus (2.2) holds true by induction.

Step 2. We show that there exists a positive integer n_0 such that T^{n_0} has a unique fixed point in E.

By $r(A) < 1$, $I - A$ is invertible, denote the inverse of $I - A$ by $(I - A)^{-1}$. Moreover, it follows from Neumann's formula that

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n = I + A + A^2 + \cdots + A^n + \cdots, \tag{2.7}$$

which implies that $(I-A)^{-1} : P \to P$ is a linear bounded mapping. It follows from $r(A) < 1$ and Gelfand's formula that there exists a positive integer n_1 and $\beta \in (r(A), 1)$ such that

$$\|A^n\| \leqslant \beta^n, \quad \forall\, n \geqslant n_1. \tag{2.8}$$

Thus for each $u \in P$, we get

$$\|A^n u\| \leqslant \|A^n\| \|u| \leqslant \beta^n \|u\|, \quad \forall\, n \geqslant n_1,$$

which implies $A^n u \overset{\|\cdot\|}{\to} \theta$ for each $u \in P$, and hence by Lemma 1.8,

$$A^n u \overset{w}{\to} \theta, \quad \forall\, u \in P. \tag{2.9}$$

Since $(I-A)^{-1}: P \to P$ is a linear bounded mapping, in analogy to (2.9), by (2.8) we obtain

$$A^n(I-A)^{-1}u \overset{w}{\to} \theta, \quad \forall\, u \in P. \tag{2.10}$$

Let $x_0 \in E_{T-C}$ and set $\{x_n\} = O(T, x_0)$, then by Step 1, there exists $u_{x_0,x_1} \in P$ such that

$$-u_{x_0,x_1} \preceq x_0 - x_1 \preceq u_{x_0,x_1}$$

and

$$-A^n u_{x_0,x_1} \preceq x_{n+1} - x_n = T^n x_1 - T^n x_0 \preceq A^n u_{x_0,x_1}, \quad \forall\, n.$$

Thus by (2.7), for each $m > n$ we have

$$-A^n(I-A)^{-1}u_{x_0,x_1} \preceq -\sum_{i=n}^{m-1} A^i u_{x_0,x_1} \preceq x_m - x_n = \sum_{i=n}^{m-1}(x_{i+1}-x_i) \preceq \sum_{i=n}^{m-1} A^i u_{x_0,x_1} \preceq A^n(I-A)^{-1}u_{x_0,x_1},$$

which together with (2.10) and Lemma 1.7 implies that

$$x_m - x_n \overset{w}{\to} \theta (m > n \to \infty), \tag{2.11}$$

i.e., $\{x_n\}$ is w-Cauchy. Note that T is Picard-complete at x_0, then there exists some $x^* \in E$ such that

$$x_n \overset{w}{\to} x^*(n \to \infty). \tag{2.12}$$

By Step 1, there exists $u_{x_0,x^*} \in P$ such that $-u_{x_0,x^*} \preceq x_0 - x^* \preceq u_{x_0,x^*}$ and

$$-A^n u_{x_0,x^*} \preceq x_n - T^n x^* = T^n x_0 - T^n x^* \preceq A^n u_{x_0,x^*},$$

which together with (2.9) and Lemma 1.7 implies that

$$x_n - T^n x^* \overset{w}{\to} \theta(n \to \infty). \tag{2.13}$$

For each $\epsilon \in \mathrm{int}P$, it follows from (2.11) and (2.13) that there exists a positive integer n_0 such that

$$-\frac{\epsilon}{2} \ll x_m - x_n \ll \frac{\epsilon}{2}, \quad \forall\, m > n \geqslant n_0, \tag{2.14}$$

and

$$-\frac{\epsilon}{2} \ll x_n - T^n x^* \ll \frac{\epsilon}{2}, \quad \forall\, n \geqslant n_0. \tag{2.15}$$

Thus by (2.14) and (2.15) we get

$$-\epsilon \ll x_m - T^{n_0}x^* = x_m - x_{n_0} + x_{n_0} - T^{n_0}x^* \ll \epsilon, \quad \forall\, m > n_0,$$

and hence

$$x_m \overset{w}{\to} T^{n_0}x^*(m \to \infty).$$

Moreover by Lemma 1.6, we get $x^* = T^{n_0}x^*$, since $\{x_n\}$ has a unique w-limit. Suppose that z is a fixed point of T^{n_0}, then by Step 1, there exists u_{z,x^*} such that $-u_{z,x^*} \preceq z - x^* \preceq u_{z,x^*}$ and

$$-A^{nn_0}u_{z,x^*} \preceq z - x^* = T^{nn_0}z - T^{nn_0}x^* \preceq A^{nn_0}u_{z,x^*}, \quad \forall\, n,$$

which together with (2.9) and Lemma 1.7 implies that $z = x^*$. Hence x^* is the unique fixed point of T^{n_0}.

Step 3. We show that x^* is the unique fixed point of T.

Note that $T^{n_0}(Tx^*) = T^{n_0+1}x^* = T(T^{n_0}x) = Tx^*$, then Tx^* is a fixed point of T^{n_0}, and hence $x^* = Tx^*$ by the uniqueness of fixed point of T^{n_0}. This shows x^* is a fixed point of T. Suppose that $z \in E$ is a fixed point of T, then z is a fixed point of T^{n_0}, and hence $z = x^*$ by the unique existence of fixed point of T^{n_0}. Hence x^* is the unique fixed point of T. □

Example 2.2. Let $E = C^1_{\mathbb{R}}[0,1]$ be endowed with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and

$$P = \{x \in E : x(t) \geqslant 0, \forall\, t \in [0,1]\},$$

where $\|x\|_\infty = \max_{t\in[0,1]} x(t)$ for each $x \in C_{\mathbb{R}}[0,1]$. Then $(E, \|\cdot\|)$ is a Banach space and P is a non-normal solid cone [1]. Let $x_0(t) \equiv \frac{1}{2}$, $D = \{x \in E : \|x\| \leqslant \frac{1}{2}\}$ and $(Tx)(t) = \int_0^t x^2(s)ds$ for each $x \in E$ and each $t \in [0,1]$. Clearly, $x_0 \in D$ and $T(D) \subset D$ since $\|Tx\| = \|Tx\|_\infty + \|(Tx)'\|_\infty \leqslant \frac{1}{2}$ for each $x \in D$.

Set $\{x_n\} = O(T, x_0)$. By induction we get

$$x_n(t) = \int_0^t x^2_{n-1}(s)ds = \frac{t^{2^n-1}}{2^{2^n}(2^2-1)^{2^{n-2}}(2^3-1)^{\alpha^{n-3}}\cdots(2^n-1)}, \quad \forall\, t \in [0,1], \quad \forall\, n \geqslant 2,$$

and so

$$\theta \preceq x_n \preceq \frac{1}{2^{2^n}(2^2-1)^{2^{n-2}}(2^3-1)^{\alpha^{n-3}}\cdots(2^n-1)}, \quad \forall\, n \geqslant 2,$$

which together with Lemma 1.7 implies that $x_n \stackrel{w}{\to} \theta$. Moreover by (i) of Remark 1.11, we know that T is Picard-complete at x_0.

For each $x, y \in D$ with $y \preceq x$ and each $t \in [0,1]$, we have

$$-\int_0^t (x(s)-y(s))ds \leqslant (Tx)(t)-(Ty)(t) = \int_0^t (x(s)-y(s))(x(s)+y(s))ds \leqslant \int_0^t (x(s)-y(s))ds,$$

and so

$$-A(x-y) \preceq Tx - Ty \preceq A(x-y), \quad \forall\, x, y \in D, \quad y \preceq x,$$

where $(Ax)(t) = \int_0^t x(s)ds$ for each $x \in E$ and each $t \in [0,1]$. This shows that $T : D \to D$ is an order-Lipschitz mapping.

For each $x \in E$ and $t \in [0,1]$, by induction we get $(A^n x)(t) \leqslant \frac{\|x\|_\infty t^n}{n!} \leqslant \frac{\|x\|}{n!}$, and so $\|A^n x\|_\infty \leqslant \frac{\|x\|}{n!}$. On the other hand, we have $\|(A^n x)'\|_\infty = \|A^{n-1}x\|_\infty \leqslant \frac{\|x\|}{(n-1)!}$ since $(A^n x)'(t) = (A^{n-1}x)(t)$. Thus $\|A^n x\| = \|A^n x\|_\infty + \|(A^n x)'\|_\infty \leqslant \frac{\|x\|}{n!} + \frac{\|x\|}{(n-1)!}$ and $\|A^n\| \leqslant \frac{1}{n!} + \frac{1}{(n-1)!}$. By Gelfand's formula, we obtain $r(A) = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n!} + \frac{1}{(n-1)!}} \leqslant \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} + \lim_{n\to\infty} \frac{1}{\sqrt[n]{(n-1)!}} = 0$. Therefore $T : D \to D$ has a unique fixed point in D by Theorem 2.1 (in fact, θ is the unique fixed point of T).

However, Theorems 1.1, 1.2, 1.3 and 1.4 are not applicable here since P is non-normal and there do not exist $A, B \in P$ or nonnegative real numbers A, B such that (1.1) is satisfied.

Remark 2.3. Theorem 2.1 implies that Theorems 1.1 and 1.2 are still valid in the case that P is non-normal, and hence Theorem 2.1 improves Theorems 1.1 and 1.2. In fact, Theorems 1.1 and 1.2 are immediate consequences of Theorem 2.1 by Remark 1.11 (ii).

In particular when E is a Banach algebra and $A, B \in P$, we have the following corollary by Theorem 2.1.

Corollary 2.4. *Let P be a solid cone of a Banach algebra $(E, \|\cdot\|)$ and $T : E \to E$ an order-Lipschitz mapping such that (1.1) is satisfied with $A, B \in P$. If $A = B$, $r(A) < 1$ and E_{T-C} is nonempty, where*

$$E_{T-C} = \{x \in E : T \text{ is Picard-complete at } x\},$$

then T has a unique fixed point $x^* \in E$. Moreover, for each $x_0 \in E_{T-C}$, we have $x_n \stackrel{w}{\to} x^*$, where $\{x_n\} = O(T, x_0)$.

Remark 2.5. It is clear that Theorem 5 in [3] is a particular case of our Corollary 2.4 with normal cones. Note that if (1.1) is satisfied with $A \in P$ and $B = \theta$ then $T : [u_0, v_0] \to E$ is nondecreasing, and hence Corollary 2.4 partially improves Theorem 1.4 since (1.2) and the nondecreasing property of T are not assumed.

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