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Article

Jensen Functional, Quasi-Arithmetic Mean and Sharp Converses of Hölder's Inequalities

Slavko Simić [1,*] and Vesna Todorčević [1,2]

[1] Mathematical Institute SANU, 11000 Belgrade, Serbia; vesna.todorcevic@fon.bg.ac.rs
[2] Department of Mathematics, Faculty of Organizational Sciences, University of Belgrade, 11000 Belgrade, Serbia
* Correspondence: ssimic@turing.mi.sanu.ac.rs

Abstract: In this article, we give sharp two-sided bounds for the generalized Jensen functional $J_n(f,g,h;\boldsymbol{p},\boldsymbol{x})$. Assuming convexity/concavity of the generating function h, we give exact bounds for the generalized quasi-arithmetic mean $A_n(h;\boldsymbol{p},\boldsymbol{x})$. In particular, exact bounds are determined for the generalized power means in terms from the class of Stolarsky means. As a consequence, some sharp converses of the famous Hölder's inequality are obtained.

Keywords: quasi-arithmetic means; power means; convex functions; Hölder's inequality

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1. Introduction

Recall that the Jensen functional $J_n(\phi;\boldsymbol{p},\boldsymbol{x})$ is defined on an interval $I \subseteq \mathbb{R}$ by

$$J_n(\phi;\boldsymbol{p},\boldsymbol{x}) := \sum_1^n p_i\phi(x_i) - \phi(\sum_1^n p_i x_i),$$

where $\phi : I \to \mathbb{R},\ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in I^n$ and $\boldsymbol{p} = \{p_i\}_1^n, \sum_1^n p_i = 1$, is a positive weight sequence.

If ϕ is a convex function on I, then the inequality

$$0 \le J_n(\phi;\boldsymbol{p},\boldsymbol{x})$$

holds for each $\boldsymbol{x} \in I^n$ and any positive weight sequence $\boldsymbol{p}$.

Jensen's inequality plays a fundamental role in many parts of mathematical analysis and applications. For example, well known $\mathcal{A}-\mathcal{G}-\mathcal{H}$ inequality, Hölder's inequality, Ky Fan inequality, etc., are proven by the help of Jensen's inequality (cf. [1–4]).

Assuming that $\boldsymbol{x} \in [a,b]^n \subset I^n$, our aim in this paper is to determine some sharp bounds for the generalized Jensen functional

$$J_n(f,g,h;\boldsymbol{p},\boldsymbol{x}) := f(\sum_1^n p_i h(x_i)) - g(h(\sum_1^n p_i x_i)),$$

for suitably chosen functions f, g and h, such that

$$c_{f,g,h}(a,b) \le J_n(f,g,h;\boldsymbol{p},\boldsymbol{x}) \le C_{f,g,h}(a,b),$$

i.e., the bounds which does not depend on $\boldsymbol{p}$ or $\boldsymbol{x}$, but only on a, b and functions f, g and h.

Our global bounds will be entirely presented in terms of elementary means.

Recall that the *mean* is a map $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, with a property

$$\min(x,y) \le M(x,y) \le \max(x,y),$$

for each $x, y \in \mathbb{R}_+$.

In order to make our results condensed and applicable, we shall use in the sequel the class of so-called Stolarsky (or extended) two-parametric mean values, defined for positive values of $x, y, x \neq y$ by the following:

$$E_{r,s}(x,y) = \begin{cases} \left(\frac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)}, & rs(r-s) \neq 0 \\ \exp\left(\frac{-1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0 \\ \left(\frac{x^s - y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0, r = 0 \\ \sqrt{xy}, & r = s = 0, \\ x, & y = x > 0. \end{cases}$$

In this form, it was introduced by Keneth Stolarsky in [5].
Most of the classical two variable means are just special cases of the class E.
For example,

$$A(x,y) = E_{1,2}(x,y) = \frac{x+y}{2}$$

is the arithmetic mean;

$$G(x,y) = E_{0,0}(x,y) = E_{-r,r}(x,y) = \sqrt{xy}$$

is the geometric mean;

$$L(x,y) = E_{0,1}(x,y) = \frac{x-y}{\log x - \log y}$$

is the logarithmic mean;

$$I(x,y) = E_{1,1}(x,y) = (x^x/y^y)^{\frac{1}{x-y}}/e$$

is the identric mean, etc.

More generally, the r-th power mean

$$A_r(x,y) = \left(\frac{x^r + y^r}{2}\right)^{1/r}$$

is equal to $E_{r,2r}(x,y)$.

Theory of Stolarsky means is very well developed, cf. [6,7] and references therein.
Some basic properties are listed in the following:
Means $E_{r,s}(x,y)$ are
a. symmetric in both parameters, i.e., $E_{r,s}(x,y) = E_{s,r}(x,y)$;
b. symmetric in both variables, i.e., $E_{r,s}(x,y) = E_{r,s}(y,x)$;
c. homogeneous of order one, that is $E_{r,s}(tx,ty) = tE_{r,s}(x,y),\ t > 0$;
d. monotone increasing in either r or s;
e. monotone increasing in either x or y; and
f. logarithmically convex in either r or s for $r,s \in \mathbb{R}_-$ and logarithmically concave for $r,s \in \mathbb{R}_+$.

Let $h : I \to J$ be a continuous and strictly monotone function on an interval $I \subset \mathbb{R}$. Then, its inverse function $h^{-1} : J \to I$ exists and generates so-called $quasi-arithmetic$ mean $\mathcal{A}_h(p,x)$, given by

$$\mathcal{A}_h(p,x) := h^{-1}\left(\sum_1^n p_i h(x_i)\right),$$

where $x = (x_1, x_2, \cdots, x_n) \in I^n$ and $p = \{p_i\}_1^n$, $\sum_1^n p_i = 1$ is a positive weight sequence.

Quasi-arithmetic means are introduced in [1] and then investigated by a plenty of researchers with most interesting results (cf. [8]). In this article, we shall give tight two-sided bounds for the difference

$$\mathcal{A}_h(\boldsymbol{p},\boldsymbol{x}) - \mathcal{A}(\boldsymbol{p},\boldsymbol{x}).$$

An important special case is the class of generalized power means $\mathcal{B}_s(\boldsymbol{p},\boldsymbol{x})$, generated by $h(\boldsymbol{x}) = \boldsymbol{x}^s,\ s \in \mathbb{R}/\{0\}$,

$$\mathcal{B}_s(\boldsymbol{p},\boldsymbol{x}) = \Big(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i^s\Big)^{1/s}.$$

It is well known fact that power means are monotone increasing in $s \in \mathbb{R}$ (cf. [1]).

Some important particular cases are

$$\mathcal{B}_{-1}(\boldsymbol{p},\boldsymbol{x}) = (\sum_1^n \boldsymbol{p}_i/\boldsymbol{x}_i)^{-1} := \mathcal{H}(\boldsymbol{p},\boldsymbol{x});$$

$$\mathcal{B}_0(\boldsymbol{p},\boldsymbol{x}) = \lim_{s\to 0} \mathcal{B}_s(\boldsymbol{p},\mathbf{x}) = \prod_1^n \boldsymbol{x}_i^{\boldsymbol{p}_i} := \mathcal{G}(\boldsymbol{p},\boldsymbol{x});$$

$$\mathcal{B}_1(\boldsymbol{p},\boldsymbol{x}) = \sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i := \mathcal{A}(\boldsymbol{p},\mathbf{x}),$$

that is, the generalized harmonic, geometric and arithmetic means, respectively.

Therefore,

$$\mathcal{H}(\boldsymbol{p},\boldsymbol{x}) \le \mathcal{G}(\boldsymbol{p},\boldsymbol{x}) \le \mathcal{A}(\boldsymbol{p},\mathbf{x}),$$

represents the celebrated $\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequalities.

Some converses of these inequalities will be given in this paper.

For arbitrary positive sequences a and b and real numbers s,t with $1/s + 1/t = 1$, the celebrated Hölder's inequalities says that

$$\sum_1^n a_i b_i \le \Big(\sum_1^n a_i^s\Big)^{1/s} \Big(\sum_1^n b_i^t\Big)^{1/t},\ s > 1;$$

and

$$\sum_1^n a_i b_i \ge \Big(\sum_1^n a_i^s\Big)^{1/s} \Big(\sum_1^n b_i^t\Big)^{1/t},\ 0 < s < 1.$$

We shall give in the sequel precise estimations of the difference

$$\sum_1^n a_i b_i - \Big(\sum_1^n a_i^s\Big)^{1/s} \Big(\sum_1^n b_i^t\Big)^{1/t},$$

and the quotient

$$\Big(\sum_1^n a_i^s\Big)^{1/s} \Big(\sum_1^n b_i^t\Big)^{1/t} / \sum_1^n a_i b_i,$$

that is,

$$\sum_1^n a_i b_i \le \Big(\sum_1^n a_i^s\Big)^{1/s} \Big(\sum_1^n b_i^t\Big)^{1/t} \le \frac{E_{s+t,s}(a,b)E_{s+t,t}(a,b)}{G^2(a,b)} \sum_1^n a_i b_i,$$

for $1/s + 1/t = 1,\ s,t > 1;\ \ a \le a_i^{1/t}/b_i^{1/s} \le b,\ i = 1,2,...,n.$

2. Results and Proofs

Our main result concerning the generalized Jensen functional $J_n(f,g,h;\boldsymbol{p},\boldsymbol{x})$ is given by the following:

Theorem 1. *Let $f : J \to \mathbb{R}$, $g : J \to \mathbb{R}$, $h : I \to J$ be continuous and eventually differentiable functions on their domains.*

For $\boldsymbol{x} \in [a,b]^n \subset I^n$, let h be convex on I and f be an increasing function on J.

Then,

$$c_{f,g,h}(a,b) := \min_{\boldsymbol{p}}[(f \circ h + g \circ h)(\boldsymbol{p}a + (1-\boldsymbol{p})b)]$$

$$\leq J_n(f,g,h;\boldsymbol{p},\boldsymbol{x}) \leq$$

$$\max_{\boldsymbol{p}}[f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b))] := C_{f,g,h}(a,b).$$

Both bounds $c_{f,g,h}(a,b)$ and $C_{f,g,h}(a,b)$ are sharp.

Proof. Since $a \leq \boldsymbol{x}_i \leq b$, there exist non-negative numbers $\lambda_i, \mu_i; \lambda_i + \mu_i = 1$, such that $\boldsymbol{x}_i = \lambda_i a + \mu_i b,\ i = 1,2,...,n$.

Hence,

$$J_n(f,g,h;\boldsymbol{p},\boldsymbol{x}) = f(\sum_1^n \boldsymbol{p}_i h(\boldsymbol{x}_i)) - g(h(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i)) = f(\sum_1^n \boldsymbol{p}_i h(\lambda_i a + \mu_i b)) - g(h(\sum_1^n \boldsymbol{p}_i(\lambda_i a + \mu_i b)))$$

$$\leq f(\sum_1^n \boldsymbol{p}_i(\lambda_i h(a) + \mu_i h(b))) - g(h(a\sum_1^n \boldsymbol{p}_i\lambda_i + b\sum_1^n \boldsymbol{p}_i\mu_i)))$$

$$= f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b)) \leq \max_{\boldsymbol{p}}[f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b))],$$

where we denoted $\sum_1^n \boldsymbol{p}_i\lambda_i := \boldsymbol{p} \in [0,1]$.

The above estimate is valid for arbitrary sequences $\boldsymbol{p}$ and $\boldsymbol{x}$. To prove its sharpness, suppose that the maxima is reached at the point $\boldsymbol{p} = \boldsymbol{p}_0$, i.e.,

$$\max_{\boldsymbol{p}}[f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b))]$$

$$= f(\boldsymbol{p}_0 h(a) + (1-\boldsymbol{p}_0)h(b)) - g(h(\boldsymbol{p}_0 a + (1-\boldsymbol{p}_0)b)) = C_{f,g,h}(a,b).$$

Then

$$J_n(f,g,h;\boldsymbol{p}_0,\boldsymbol{x}_0) = C_{f,g,h}(a,b),$$

where

$$\boldsymbol{p}_0 = (\boldsymbol{p}_0, \boldsymbol{p}_2, ..., \boldsymbol{p}_n),\ \boldsymbol{x}_0 = (a,b,...,b).$$

On the other hand, since h is a convex function on I, by Jensen's inequality we get

$$\sum_1^n \boldsymbol{p}_i h(\boldsymbol{x}_i) \geq h(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i).$$

Because f is an increasing function, it follows that

$$J_n(f,g,h;\boldsymbol{p},\boldsymbol{x}) = f(\sum_1^n \boldsymbol{p}_i h(\boldsymbol{x}_i)) - g(h(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i)) \geq f(h(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i)) - g(h(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i))$$

$$= f(h(\boldsymbol{p}a + (1-\boldsymbol{p})b)) - g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b)) \geq \min_{\boldsymbol{p}}[(f \circ h + g \circ h)(\boldsymbol{p}a + (1-\boldsymbol{p})b)] := c_{f,g,h}(a,b).$$

A simple analysis of the constant $c_{f,g,h}(a,b)$ reveals the next: if minima of the function $(f \circ h + g \circ h)(t)$ exists for $t = t_0 \in [a,b]$, then $c_{f,g,h}(a,b) = (f \circ h + g \circ h)(t_0)$, taken for $\boldsymbol{p} = \boldsymbol{p}_0 = (b - t_0)/(b-a)$.

Otherwise, we have that $c_{f,g,h}(a,b) = \min\{(f \circ h + g \circ h)(a), (f \circ h + g \circ h)(b)\}$.

Those results are evidently sharp, since

$$J_n(f,g,h;\boldsymbol{p},\boldsymbol{x}_0) = c_{f,g,h}(a,b),$$

with $\boldsymbol{x}_0 = (t_0,...,t_0)$, $\boldsymbol{x}_0 = (a,...,a)$ or $\boldsymbol{x}_0 = (b,...,b)$, respectively. □

Theorem 1 with its variants (a decreasing function f, concave function h) is the source of a plenty of interesting inequalities. Further investigations are left to the reader.

Sometimes, it is a difficult problem to evaluate exact maxima in this theorem.

For this cause, we shall give in the sequel two estimations of $J_n(f,g,h;\boldsymbol{p},\mathbf{x})$ with the unique maxima, which could be easily calculated.

Theorem 2. *Under the conditions of Theorem 1, assume firstly that f is a convex function on J. Then,*

$$J_n(f,g,h;\boldsymbol{p},\mathbf{x}) \le \max_{\boldsymbol{p}}[\boldsymbol{p}(f\circ h)(a) + (1-\boldsymbol{p})(f\circ h)(b) - (g\circ h)(\boldsymbol{p}a + (1-\boldsymbol{p})b)].$$

Assuming that $g\circ h$ is a concave function, we obtain

$$J_n(f,g,h;\boldsymbol{p},\mathbf{x}) \le \max_{\boldsymbol{p}}[f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - (\boldsymbol{p}(g\circ h)(a) + (1-\boldsymbol{p})(g\circ h)(b))].$$

Now, both maxima can be easily determined by the standard technique.

Proof. By Theorem 1, we know that there exists $\boldsymbol{p} \in [0,1]$ such that

$$J_n(f,g,h;\boldsymbol{p},\mathbf{x}) \le f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b)).$$

If additionally f is convex on J, then

$$f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) \le \boldsymbol{p}(f\circ h)(a) + (1-\boldsymbol{p})(f\circ h)(b).$$

Hence,

$$\begin{aligned} J_n(f,g,h;\boldsymbol{p},\mathbf{x}) &\le \boldsymbol{p}(f\circ h)(a) + (1-\boldsymbol{p})(f\circ h)(b) - (g\circ h)(\boldsymbol{p}a + (1-\boldsymbol{p})b)) \\ &\le \max_{\boldsymbol{p}}[\boldsymbol{p}(f\circ h)(a) + (1-\boldsymbol{p})(f\circ h)(b) - (g\circ h)(\boldsymbol{p}a + (1-\boldsymbol{p})b))]. \end{aligned}$$

Similarly, if $g\circ h$ is a concave function on J, we have

$$g(h(\boldsymbol{p}a + (1-\boldsymbol{p})b) = (g\circ h)(\boldsymbol{p}a + (1-\boldsymbol{p})b) \ge \boldsymbol{p}(g\circ h)(a) + (1-\boldsymbol{p})(g\circ h)(b),$$

and

$$J_n(f,g,h;\boldsymbol{p},\mathbf{x}) \le \max_{\boldsymbol{p}}[f(\boldsymbol{p}h(a) + (1-\boldsymbol{p})h(b)) - (\boldsymbol{p}(g\circ h)(a) + (1-\boldsymbol{p})(g\circ h)(b))].$$

□

An important special case is the converse of Jensen's inequality.

Theorem 3. *Let ϕ be a convex function on $I \subset \mathbb{R}$ and, for $[\xi,\eta] \subset I$, let $\boldsymbol{x} \in [\xi,\eta]^n$. Then,*

$$0 \le J_n(\phi;\boldsymbol{p},\mathbf{x}) \le \max_{\boldsymbol{p}}[\boldsymbol{p}\phi(\xi) + (1-\boldsymbol{p})\phi(\eta) - \phi(p\xi + (1-\boldsymbol{p})\eta)] := T_\phi(\xi,\eta).$$

If ϕ is a concave function, then

$$0 \le -J_n(\phi;\boldsymbol{p},\mathbf{x}) \le \max_{\boldsymbol{p}}[\phi(\boldsymbol{p}\xi + (1-\boldsymbol{p})\eta) - (\boldsymbol{p}\phi(\xi) + (1-\boldsymbol{p})\phi(\eta))] = -T_\phi(\xi,\eta).$$

The constant $T_\phi(\xi,\eta)$ is sharp since there exist sequences $\boldsymbol{p}_0, \boldsymbol{x}_0$ such that

$$J_n(\phi;\boldsymbol{p}_0,\boldsymbol{x}_0) = T_\phi(\xi,\eta).$$

Proof. This is a simple consequence of Theorem 1 obtained for $f(\boldsymbol{x}) = g(\boldsymbol{x}) = \boldsymbol{x}$; $h = \phi$. If ϕ is a concave function, then $-\phi$ is convex and the proof follows from the first part of this theorem. □

In this case, the bound $T_\phi(\xi,\eta)$ can be explicitly calculated.

Theorem 4. *For a differentiable convex mapping ϕ, we have that*

$$T_\phi(\xi,\eta) = \frac{\phi(\eta)-\phi(\xi)}{\eta-\xi}\Theta_\phi(\xi,\eta) + \frac{\eta\phi(\xi)-\xi\phi(\eta)}{\eta-\xi} - \phi(\Theta_\phi(\xi,\eta)),$$

where $\Theta_\phi(\xi,\eta)$ is the Lagrange mean value of numbers ξ and η, defined by

$$\Theta_\phi(\xi,\eta) := (\phi')^{-1}\Big(\frac{\phi(\eta)-\phi(\xi)}{\eta-\xi}\Big).$$

The function T_ϕ is positive and symmetric, i.e., $T_\phi(\xi,\eta) = T_\phi(\eta,\xi)$ and $\lim_{\eta\to\xi} T_\phi(\xi,\eta) = 0$.

Proof. If the maximum is taken at the point $\boldsymbol{p} = \boldsymbol{p}_0$, by the standard technique we get

$$\phi'(\boldsymbol{p}_0\xi + (1-\boldsymbol{p}_0)\eta)(\xi-\eta) = \phi(\xi)-\phi(\eta),$$

that is,

$$\boldsymbol{p}_0\xi + (1-\boldsymbol{p}_0)\eta = \Theta_\phi(\xi,\eta).$$

Therefore,

$$\boldsymbol{p}_0 = \frac{\Theta_\phi(\xi,\eta)-\eta}{\xi-\eta};\ 1-\boldsymbol{p}_0 = \frac{\xi-\Theta_\phi(\xi,\eta)}{\xi-\eta},$$

and

$$\max_{\boldsymbol{p}}[\boldsymbol{p}\phi(\xi) + (1-\boldsymbol{p})\phi(\eta) - \phi(\boldsymbol{p}\xi + (1-\boldsymbol{p})\eta)] = \boldsymbol{p}_0\phi(\xi) + (1-\boldsymbol{p}_0)\phi(\eta) - \phi(\boldsymbol{p}_0\xi + (1-\boldsymbol{p}_0)\eta)$$

$$= \frac{\Theta_\phi(\xi,\eta)-\eta}{\xi-\eta}\phi(\xi) + \frac{\xi-\Theta_\phi(\xi,\eta)}{\xi-\eta}\phi(\eta) - \phi(\Theta_\phi(\xi,\eta)) = T_\phi(\xi,\eta).$$

□

Now, some important inequalities concerning quasi-arithmetic mean can be easily obtained from Theorem 1 by putting $f = g = h^{-1}$. Nevertheless, in order to avoid unnecessary monotonicity issues, we turn another way.

Our main result is contained in the following:

Theorem 5. *For $a \le \boldsymbol{x}_i \le b$, $i = 1,2,...,n$; $a,b \in I$, let $h : I \to J$ be continuous and strictly monotone function and assume that $h^{-1} : J \to I$ is convex. Then,*

$$0 \le \mathcal{A}(\boldsymbol{p},\boldsymbol{x}) - \mathcal{A}_h(\boldsymbol{p},\boldsymbol{x}) \le T_{h^{-1}}(h(a),h(b)),$$

where the constant $T_\phi(\xi,\eta)$ is defined in Theorems 3 and 4.

If h^{-1} is a concave function, then

$$0 \le \mathcal{A}_h(\boldsymbol{p},\boldsymbol{x}) - \mathcal{A}(\boldsymbol{p},\boldsymbol{x}) \le -T_{h^{-1}}(h(a),h(b)).$$

Proof. We shall give a simple proof of this theorem.

Namely, since h^{-1} is a convex function, applying the first part of Theorem 3 with $\phi = h^{-1}$, we obtain

$$0 \le \sum_1^n \boldsymbol{p}_i h^{-1}(\boldsymbol{x}_i) - h^{-1}(\sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i) \le T_{h^{-1}}(a,b).$$

Now, by changing variables $\boldsymbol{x}_i \to h(\boldsymbol{x}_i)$ $i = 1,2,...,n$, we get $h^{-1}(\boldsymbol{x}_i) \to h^{-1}\circ h(\boldsymbol{x}_i) = \boldsymbol{x}_i$ and $a \to h(a)$, $b \to h(b)$.

Hence, $h(a) \le h(\boldsymbol{x}_i) \le h(b)$ or $h(b) \le h(\boldsymbol{x}_i) \le h(a)$ depending on the monotonicity of h. However, since $T_\phi(\xi,\eta)$ is symmetric in variables, we finally get

$$0 \le \sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i - h^{-1}(\sum_1^n \boldsymbol{p}_i h(\boldsymbol{x}_i)) \le T_{h^{-1}}(h(a), h(b)).$$

The second part of this theorem can be proved along the same lines. □

The most striking example of quasi-arithmetic means is the class of generalized power means $\mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x})$, generated by $h(x) = x^s,\ h^{-1}(x) = x^{1/s},\ s \in \mathbb{R}/\{0\}$, i.e.,

$$\mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x}) = \Big(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i^s\Big)^{1/s}.$$

As an application of Theorem 5, we shall estimate the difference $\mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{A}(\boldsymbol{p}, \boldsymbol{x})$.

Theorem 6. *Let $a \le \boldsymbol{x}_i \le b,\ i = 1, 2, ..., n;\ 0 < a < b$.*
Then,

$$0 \le \mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) \le \frac{s-1}{s}\Big(E_{s,1}(a,b) - \frac{G^2(a,b)}{E_{s,s-1}(a,b)}\Big),\ s > 1;$$

$$0 \le \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x}) \le \frac{1-s}{s}(E_{1,s}(a,b) - E_{1-s,-s}(a,b)),\ 0 < s < 1;$$

$$0 \le \mathcal{A}(\boldsymbol{p}, \mathbf{x}) - \mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x}) \le \frac{s-1}{s}(E_{1-s,-s}(a,b) - E_{1,s}(a,b)),\ s < 0.$$

Proof. Let $h(\boldsymbol{x}) = \boldsymbol{x}^s,\ h^{-1}(\boldsymbol{x}) = \boldsymbol{x}^{1/s}, s \in \mathbb{R}/\{0\}$.

If $s > 1$ then h^{-1} is a concave function on $\mathbb{R}^+$. Hence, by the second part of Theorem 5, we get

$$0 \le \mathcal{B}_s(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) \le -T_{\boldsymbol{x}^{1/s}}(a^s, b^s).$$

Applying the result from Theorem 4, a simple calculation gives

$$\Theta_{\boldsymbol{x}^{1/s}}(a^s, b^s) = \Big(\frac{b^s - a^s}{s(b-a)}\Big)^{s/(s-1)} = E^s_{s,1}(a,b) = \frac{b^s - a^s}{s(b-a)} E_{s,1}(a,b).$$

Hence,

$$-T_{\boldsymbol{x}^{1/s}}(a^s, b^s) = (\Theta(\cdot))^{1/s} - \frac{b-a}{b^s - a^s}\Theta(\cdot) - \frac{ab^s - ba^s}{b^s - a^s}$$

$$= E_{s,1}(a,b) - \frac{1}{s}E_{s,1}(a,b) - \frac{ab(b^{s-1} - a^{s-1})}{b^s - a^s} = \frac{s-1}{s}\Big(E_{s,1}(a,b) - \frac{G^2(a,b)}{E_{s,s-1}(a,b)}\Big).$$

In cases $0 < s < 1$ or $s < 0$, one should apply the first part of Theorem 5, since $h^{-1} = x^{1/s}$ is convex on $\mathbb{R}^+$. Proceeding as above, the result follows. □

As a consequence, we obtain some converses of the $\mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{G}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{H}(\boldsymbol{p}, \boldsymbol{x})$ inequality.

Corollary 1. *Let $a \le \boldsymbol{x}_i \le b,\ i = 1, 2, ..., n;\ 0 < a < b$.*
Then,

$$0 \le \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{H}(\boldsymbol{p}, \mathbf{x}) \le 2(A(a,b) - G(a,b)).$$

Proof. Putting $s = -1$, we get

$$0 \le \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{B}_{-1}(\boldsymbol{p}, \mathbf{x}) = \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{H}(\boldsymbol{p}, \mathbf{x})$$

$$\le 2(E_{2,1}(a,b) - E_{1,-1}(a,b)) = 2(A(a,b) - G(a,b)).$$

□

Corollary 2. *Let $a \le \boldsymbol{x}_i \le b,\ i = 1, 2, ..., n;\ 0 < a < b$.*
Then,

$$0 \le \mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{G}(\boldsymbol{p}, \mathbf{x}) \le L(a,b) \log \frac{L(a,b) I(a,b)}{G^2(a,b)}.$$

Proof. We have,

$$\mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{G}(\boldsymbol{p}, \boldsymbol{x}) = \lim_{s \to 0} (\mathcal{A}(\boldsymbol{p}, \boldsymbol{x}) - \mathcal{B}_s(\boldsymbol{p}, \mathbf{x}))$$

$$\le \lim_{s \to 0} \Big(\frac{1-s}{s} (E_{1,s}(a,b) - E_{1-s,-s}(a,b)) \Big)$$

$$= L(a,b) \log \frac{L(a,b) I(a,b)}{G^2(a,b)}.$$

□

The sequences $\boldsymbol{p}$ and $\boldsymbol{x}$ in Theorem 6 are arbitrary. Specializing a little bit, we obtain sharp converses of slightly generalized Hölder's inequalities.

Theorem 7. *Let $\{t_i\}_1^n$, $\{u_i\}_1^n$, $\{v_i\}_1^n$ be any sequences of positive real numbers with $a \le u_i v_i^{1-t} \le b$ for some constants $0 < a < b$ and $1/s + 1/t = 1$ for some $s, t \in \mathbb{R}$.*
Then,

$$0 \le \Big(\sum_1^n t_i u_i^s \Big)^{1/s} \Big(\sum_1^n t_i v_i^t \Big)^{1/t} - \sum_1^n t_i u_i v_i \le C_s(a,b) \sum_1^n t_i v_i^t,$$

with

$$C_s(a,b) = \frac{s-1}{s} \Big(E_{s,1}(a,b) - \frac{G^2(a,b)}{E_{s,s-1}(a,b)} \Big),$$

and $s > 1$;

$$0 \le \sum_1^n t_i u_i v_i - \Big(\sum_1^n t_i u_i^s \Big)^{1/s} \Big(\sum_1^n t_i v_i^t \Big)^{1/t} \le D_s(a,b) \sum_1^n t_i v_i^t,$$

where

$$D_s(a,b) = \frac{1-s}{s} \Big(E_{s,1}(a,b) - E_{1-s,-s}(a,b) \Big),$$

and $0 < s < 1$.

Proof. Changing variables

$$\boldsymbol{p}_i = t_i v_i^t / \sum_i^n t_i v_i^t;\ \ \boldsymbol{x}_i = u_i v_i^{1-t},\ i = 1, 2, ..., n,$$

yields

$$\boldsymbol{p}_i \boldsymbol{x}_i = t_i u_i v_i / \sum_1^n t_i v_i^t;$$

$$\boldsymbol{p}_i \boldsymbol{x}_i^s = t_i v_i^t (u_i v_i^{1-t})^s / \sum_1^n t_i v_i^t = t_i u_i^s v_i^{s+t-st} / \sum_1^n t_i v_i^t = t_i u_i^s / \sum_1^n t_i v_i^t.$$

Now, applying Theorem 6 for $s > 1$, we get

$$0 \le \mathcal{B}_s(\boldsymbol{p},\boldsymbol{x}) - \mathcal{A}(\boldsymbol{p},\boldsymbol{x}) = (\sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i^s)^{1/s} - \sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i$$

$$= \frac{(\sum_1^n t_i u_i^t)^{1/s}}{(\sum_1^n t_i v_i^t)^{1/s}} - \frac{\sum_1^n t_i u_i v_i}{\sum_1^n t_i v_i^t} \le \frac{s-1}{s}\Big(E_{s,1}(a,b) - \frac{G^2(a,b)}{E_{s,s-1}(a,b)}\Big),$$

and the result clearly follows by multiplying both sides with $\sum_1^n t_i v_i^t$.

Applying the same procedure in the case $0 < s < 1$, we obtain the second part of this theorem. □

Finally, we prove another sharp converses of Hölder's inequalities. For this cause, we shall estimate firstly the expression

$$\mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) := (\sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i^s)^{1/s}(\sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i^{-t})^{1/t}, \ 1/s + 1/t = 1, \ s,t \in \mathbb{R}.$$

Lemma 1. *Let $a \le \boldsymbol{x}_i \le b, \ i = 1,2,...,n$ for some $0 < a < b$.*

If $s > 1$, we have

$$1 \le \mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) \le \frac{E_{s,s+t}(a,b)E_{t,s+t}(a,b)}{G^2(a,b)},$$

and

$$\frac{E_{s,s+t}(a,b)E_{t,s+t}(a,b)}{G^2(a,b)} \le \mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) \le 1,$$

for $0 < s < 1$.

Proof. Following the method from the proof of Theorem 1, we get

$$\boldsymbol{x}_i^s = \lambda_i a^s + \mu_i b^s, \ \lambda_i + \mu_i = 1, \ i = 1,2,...,n.$$

If $s > 1$, then also $t > 1$, hence the function $\boldsymbol{x}^{-t/s}$ is convex.

Therefore,

$$\boldsymbol{x}_i^{-t} = (\boldsymbol{x}_i^s)^{-t/s} = (\lambda_i a^s + \mu_i b^s)^{-t/s} \le \lambda_i (a^s)^{-t/s} + \mu_i (b^s)^{-t/s} = \lambda_i a^{-t} + \mu_i b^{-t},$$

and

$$\mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) = (\sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i^s)^{1/s}(\sum_1^n \boldsymbol{p}_i\boldsymbol{x}_i^{-t})^{1/t}$$

$$\le (a^s\sum_1^n \boldsymbol{p}_i\lambda_i + b^s\sum_1^n \boldsymbol{p}_i\mu_i)^{1/s}(a^{-t}\sum_1^n \boldsymbol{p}_i\lambda_i + b^{-t}\sum_1^n \boldsymbol{p}_i\mu_i)^{1/t}$$

$$= (pa^s + qb^s)^{1/s}(pa^{-t} + qb^{-t})^{1/t},$$

where we put

$$\sum_1^n \boldsymbol{p}_i\lambda_i := p, \ \sum_1^n \boldsymbol{p}_i\mu_i := q; \ p + q = 1.$$

Therefore, it follows that

$$\mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) \le \max_{\boldsymbol{p}}[(pa^s + qb^s)^{1/s}(pa^{-t} + qb^{-t})^{1/t}]$$

$$= (\boldsymbol{p}_0 a^s + q_0 b^s)^{1/s}(\boldsymbol{p}_0 a^{-t} + q_0 b^{-t})^{1/t}.$$

By the standard technique we obtain that this maxima satisfy the equation

$$\frac{s(\boldsymbol{p}_0 a^s + q_0 b^s)}{a^s - b^s} = \frac{-t(\boldsymbol{p}_0 a^{-t} + q_0 b^{-t})}{a^{-t} - b^{-t}},$$

that is,

$$\boldsymbol{p}_0 = \frac{1}{s+t}\Big(\frac{sb^s}{b^s - a^s} - \frac{ta^t}{b^t - a^t}\Big);\ q_0 = \frac{1}{s+t}\Big(\frac{tb^t}{b^t - a^t} - \frac{sa^s}{b^s - a^s}\Big).$$

Henceforth,

$$\boldsymbol{p}_0 a^{-t} + q_0 b^{-t} = \frac{s}{s+t}\frac{a^{-t}b^s - a^s b^{-t}}{b^s - a^s} = \frac{s}{s+t}\frac{(ab)^{-t}(b^{s+t} - a^{s+t})}{b^s - a^s},$$

and

$$\boldsymbol{p}_0 a^s + q_0 b^s = \frac{t}{s+t}\frac{b^{s+t} - a^{s+t}}{b^t - a^t}.$$

Therefore,

$$(\boldsymbol{p}_0 a^{-t} + q_0 b^{-t})^{1/t} = E_{s+t,s}(a,b)/G^2(a,b);$$

$$(\boldsymbol{p}_0 a^s + q_0 b^s)^{1/s} = E_{s+t,t}(a,b),$$

and we finally obtain

$$\max_{\boldsymbol{p}}[(pa^s + qb^s)^{1/s}(pa^{-t} + qb^{-t})^{1/t}] = (\boldsymbol{p}_0 a^s + q_0 b^s)^{1/s}(\boldsymbol{p}_0 a^{-t} + q_0 b^{-t})^{1/t}$$

$$= \frac{E_{s+t,s}(a,b)E_{s+t,t}(a,b)}{G^2(a,b)}.$$

On the other hand, by the monotonicity in s of $\mathcal{B}_s$, we get

$$1 \le \frac{\mathcal{B}_s(\boldsymbol{p},\boldsymbol{x})}{\mathcal{B}_{-t}(\boldsymbol{p},\boldsymbol{x})} = \mathcal{F}_{s,t}(\boldsymbol{p},\mathbf{x}),$$

since $s > -t$.

In the case $0 < s < 1$, we have that $t < 0$. Therefore,

$$0 > st = s + t,$$

and

$$\mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) \le 1,$$

since $s < -t$.

Additionally, $-t/s > 1$, hence $x^{-t/s}$ is a convex function.

Therefore,

$$\boldsymbol{x}_i^{-t} = (\boldsymbol{x}_i^s)^{-t/s} = (\lambda_i a^s + \mu_i b^s)^{-t/s} \le \lambda_i (a^s)^{-t/s} + \mu_i (b^s)^{-t/s} = \lambda_i a^{-t} + \mu_i b^{-t}.$$

However, because the exponent $1/t$ is negative in this case, we obtain

$$\mathcal{F}_{s,t}(\boldsymbol{p},\boldsymbol{x}) = (\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i^s)^{1/s}(\sum_1^n \boldsymbol{p}_i \boldsymbol{x}_i^{-t})^{1/t}$$

$$\ge (a^s \sum_1^n \boldsymbol{p}_i \lambda_i + b^s \sum_1^n \boldsymbol{p}_i \mu_i)^{1/s}(a^{-t}\sum_1^n \boldsymbol{p}_i \lambda_i + b^{-t}\sum_1^n \boldsymbol{p}_i \mu_i)^{1/t}$$

$$= (\boldsymbol{p}a^s + qb^s)^{1/s}(\boldsymbol{p}a^{-t} + qb^{-t})^{1/t}.$$

Therefore, we get

$$\mathcal{F}_{s,t}(\boldsymbol{p},\mathbf{x}) \geq \min_{\boldsymbol{p}}[(\boldsymbol{p}a^s + qb^s)^{1/s}(\boldsymbol{p}a^{-t} + qb^{-t})^{1/t}],$$

and, proceeding as above, the second part of this theorem follows. □

We are now able to formulate our main result.

Theorem 8. *Let $\{t_i\}_1^n$, $\{u_i\}_1^n$, $\{v_i\}_1^n$ be arbitrary sequences of positive numbers with $a \leq u_i^{1/t}/v_i^{1/s} \leq b$ for some constants $0 < a < b$ and $1/s + 1/t = 1$; $s,t \in \mathbb{R}$.*

For $s > 1$, we have

$$\sum_1^n t_i u_i v_i \leq \Big(\sum_1^n t_i u_i^s\Big)^{1/s}\Big(\sum_1^n t_i v_i^t\Big)^{1/t} \leq \frac{E_{s,s+t}(a,b)E_{t,s+t}(a,b)}{G^2(a,b)}\sum_1^n t_i u_i v_i,$$

and

$$\frac{E_{s,s+t}(a,b)E_{t,s+t}(a,b)}{G^2(a,b)}\sum_1^n t_i u_i v_i \leq \Big(\sum_1^n t_i u_i^s\Big)^{1/s}\Big(\sum_1^n t_i v_i^t\Big)^{1/t} \leq \sum_1^n t_i u_i v_i,$$

for $0 < s < 1$.

Proof. Changing variables

$$\boldsymbol{p}_i = t_i u_i v_i / \sum_1^n t_i u_i v_i; \;\; \boldsymbol{x}_i = u_i^{1/t} v_i^{-1/s}, \;\; i = 1,2,...,n,$$

we get

$$\boldsymbol{p}_i \boldsymbol{x}_i^s = t_i u_i v_i (u_i^{1/t} v_i^{-1/s})^s / \sum_1^n t_i u_i v_i = t_i u_i^{1+s/t} / \sum_1^n t_i u_i v_i = t_i u_i^s / \sum_1^n t_i u_i v_i;$$

$$\boldsymbol{p}_i \boldsymbol{x}_i^{-t} = t_i u_i v_i (u_i^{1/t} v_i^{-1/s})^{-t} / \sum_1^n t_i u_i v_i = t_i v_i^{1+t/s} / \sum_1^n t_i u_i v_i = t_i v_i^t / \sum_1^n t_i u_i v_i,$$

and

$$\mathcal{F}_{s,t}(t,u,v) = \Big(\frac{\sum_1^n t_i u_i^s}{\sum_1^n t_i u_i v_i}\Big)^{1/s}\Big(\frac{\sum_1^n t_i v_i^t}{\sum_1^n t_i u_i v_i}\Big)^{1/t} = \frac{(\sum_1^n t_i u_i^s)^{1/s}(\sum_1^n t_i v_i^t)^{1/t}}{\sum_1^n t_i u_i v_i}.$$

Now, an application of Lemma 1 gives the result. □

3. Conclusions

In this article, we give further development of our results from [3]. Sharp two-sided bounds are explicitly determined for the generalized Jensen functional $J_n(f,g,h;\boldsymbol{p},\boldsymbol{x})$ and, consequently, for Jensen's inequality and quasi-arithmetic means. Exact converses of $\mathcal{A} - \mathcal{G} - \mathcal{H}$ inequalities and some forms of Hölder's inequalities are also given. Since Theorem 1 achieved its definite form with very mild conditions posed on the generating functions f,g and h, there remains a lot of work to apply its results in different areas of mathematics.

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