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A note on traces of some holomorphic spaces on polyballs

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Abstract. We study a new trace problem for functions holomorphic on polyballs which generalize a known diagonal map problem for polydisk. Also, we give descriptions of traces for several concrete functional classes on polyballs defined with the help of area operator or Bergman metric ball.

1. Introduction

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $\mathbf{B} = \{z \in \mathbb{C}^n | |z| < 1\}$. The boundary of $\mathbf{B}$ will be denoted by $\mathbf{S}$, $\mathbf{S} = \{z \in \mathbb{C}^n | |z| = 1\}$.

As usual, we denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$. For every function $f \in H(\mathbf{B})$ having a series expansion $f(z) = \sum_{|k|\geq 0} a_k z^k$,

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we define the operator of fractional differentiation by

$$D^{\alpha} f(z) = \sum_{|k|\geq 0} (|k|+1)^{\alpha} a_k z^k,$$

where α is any real number. It is obvious that for any α, D^{α} is an operator acting from $H(\mathbf{B})$ to $H(\mathbf{B})$.

For a fixed $\alpha > 1$ let $\Gamma_{\alpha}(\xi) = \{z \in \mathbf{B} : |1-\overline{\xi}z| < \alpha(1-|z|)\}$ be the admissible approach region with vertex at $\xi \in \mathbf{S}$.

Let dv denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B}) = 1$, and let $d\sigma$ denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S}) = 1$.

For $\alpha > -1$ the weighted Lebesgue measure dv_{α} is defined by

$$dv_{\alpha} = c_{\alpha}(1-|z|^2)^{\alpha} dv(z), \tag{1}$$

where

$$c_{\alpha} = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \tag{2}$$

is a normalizing constant so that $v_{\alpha}(\mathbf{B}) = 1$ (see [18]).

Let also $d\widetilde{v}_{\beta}(z) = dv_{\beta}(z_1)\dots dv_{\beta}(z_m) = (1-|z_1|^2)^{\beta}\cdots(1-|z_n|^2)^{\beta} dv(z_1)\cdots dv(z_n)$. For $z \in \mathbf{B}$ and $r > 0$ the set $\mathcal{D}(z,r) = \{w \in \mathbf{B} : \beta(z,w) < r\}$ where β is a Bergman metric on $\mathbf{B}$, $\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$ is called the Bergman metric ball at z (see [18]).

For $\alpha > -1$ and $p > 0$ the weighted Bergman space A^p_{α} consists of holomorphic functions f in $L^p(\mathbf{B}, dv_{\alpha})$, that is,

$$A^p_{\alpha} = L^p(\mathbf{B}, dv_{\alpha}) \cap H(\mathbf{B}).$$

See [5] and [18] for more details of weighted Bergman spaces.

Let m, $m > 1$ be a natural number, $M \subset \mathbb{C}^n$ and $K \subset \mathbb{C}^{mn}$, $C^{mn} = C^n \times \cdots \times C^n$, be a hyper surface. Let $X(M)$ be a class of functions on M, $Y(K)$ the same. We say $Trace Y(M^m) = X(M)$, $K = M^m$, $M^m = M \times \cdots \times M$, if for any $f \in Y(M^m)$, $f(w,\dots,w) \in X(M)$, $w \in M$, and for any $g \in X(M)$, there exist a function $f \in Y(K)$ such that $f(w,\dots,w) = g(w)$, $w \in M$. Traces of various functional spaces in $\mathbb{R}^n$ were described in [9] and [17]. In polydisk this problem is also known as a problem of diagonal map (see [5] and references there).

The intention of this paper is to consider the following natural Trace problem for polyballs. Let M be a unit ball and let K be a polyball (product of m balls) in definition we gave above. Let further $H(\mathbf{B}\times\cdots\times\mathbf{B})$ be a space of all holomorphic functions by each z_j, $z_j \in B$, $j = 1,\dots,m$: $f(z_1,\dots,z_m)$. Let further Y be a subspace of $H(\mathbf{B}\times\cdots\times\mathbf{B})$.

The question we would like to study and solve in this work is the following: Find the complete description of Trace Y in a sense of our definition for several concrete functional classes. We observe that for $n = 1$ this problem completely coincide with the well-known problem of diagonal map. The last problem of description of diagonal of various subspaces of $H(\mathbf{D}^n)$ of spaces of all holomorphic functions in the polydisk was studied by many authors before (see [5], [7], [12], [13], [16] and references there). With the help of area operator and Bergman metric ball in $\mathbf{B}$ we introduce new holomorphic functional classes on polyballs and describe completely their Traces via classical Bergman spaces in the unit ball $\mathbf{B}$ of $\mathbb{C}^n$.

In our previous paper (see [15]), we completely described traces of weighted Bergman classes on polyballs for all values of $p \in (0, \infty)$ and traces of some analytic Bloch type spaces on polyballs expanding known theorems on diagonal map in polydisk (see [5], [12] and references there). Some results of this paper are new even for $n = 1$ (polydisk case). Main results of this paper will be proved in next section. In the final section, we consider related estimates for spaces defined with the help of fractional derivatives. Basic properties of a known so-called r-lattice in the Bergman metric that can be found in [18] and estimates of expanded Bergman projection in the unit ball are essential for our proofs.

Trace theorems even for $n = 1$ (case of polydisk) have numerous applications in the theory of holomorphic functions (see for example [2], [5], [14]).

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

2. The description of traces of analytic functional spaces in polyballs based on area operator and Bergman metric ball and the action of expanded Bergman projection

Proofs of all our theorems in this section are heavily based on properties of r-lattice $\{a_k\}$ in a Bergman metric (see for example [18]). In particular we will use systematically the following lemmas.

Lemma A. [18] *There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in $\mathbf{B}$ with the following properties:*

(1) $\mathbf{B} = \bigcup_k \mathcal{D}(a_k, r)$;

(2) *The sets* $\mathcal{D}(a_k, \frac{r}{4})$ *are mutually disjoint;*

(3) *Each point* $z \in \mathbf{B}$ *belongs to at most* N *of the sets* $\mathcal{D}(a_k, 4r)$

Lemma B. [18] *For each* $r > 0$ *there exists a constant* $C_r > 0$ *such that*

$$C_r^{-1} \le \frac{1-|a|^2}{1-|z|^2} \le C_r, \qquad C_r^{-1} \le \frac{1-|a|^2}{|1-\langle z, a\rangle|} \le C_r,$$

for all a *and* z *such that* $\beta(a,z) < r$. *Moreover, if* r *is bounded above, then we may choose* C_r *independent of* r.

We will need also:

Lemma C. [8] *Let* $\beta > 0$ *and* $p > 0$. *Let* $\{A_j\}_0^\infty$ *be a positive sequence and* $\sum_{n=1}^\infty 2^{-n\beta} A_n^p < \infty$. *Then*

$$\sum_{n=1}^{\infty} 2^{-n\beta} A_n^p \le C \left(A_0^p + \sum_{n=1}^{\infty} 2^{-n\beta} |A_n - A_{n-1}|^p \right).$$

Bergman classes on polyballs $A^p(\mathbf{B}^m, dv_{\alpha_1} \cdots dv_{\alpha_m})$ consists of functions f in $H(\mathbb{B}^m)$, such that,

$$\int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |f(z_1, \dots, z_m)|^p (1-|z_1|)^{\alpha_1} \cdots (1-|z_m|)^{\alpha_m} dv(z_1) \cdots dv(z_m) < \infty.$$

We need also the following theorem from our paper [15] where the description of traces of Bergman classes in polyballs were given.

Theorem A. (1) *Suppose* $1 \le p \le \infty$ *and* $s_1, \dots, s_m > -1$. *Put* $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. *Then there are bounded operators* $S : A^p(\mathbf{B}, dv_t) \to A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m})$, *and* $R : A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m}) \to A^p(\mathbf{B}, dv_t)$ *such that* $(Sf)(z, \dots, z) = f(z)$ *and* $(Rg)(z) = g(z, \dots, z)$ *for all* $f \in A^p(\mathbf{B}, dv_t)$, *all* $g \in A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m})$ *and all* $z \in \mathbf{B}$. *In other words, the* $Trace A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m}) = A^p(\mathbf{B}, dv_t)$.

(2) *Let* $0 < p \le 1$, $s_1, \dots, s_m > -1$, $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. *Then* $Trace A^p(\mathbf{B}^m, dv_{s_1}, \dots, dv_{s_m}) = A^p(\mathbf{B}, dv_t)$.

In this section, we will give the complete description of traces of the following spaces of holomorphic functions on polyballs $\mathbb{B}^m = \mathbb{B} \times \cdots \times \mathbb{B}$ defined with the help of area operator and Bergman metric ball with some restrictions on parameters.

$$M_\alpha^p = \left\{ f \in H(\mathbb{B}^m) : \int_{\mathbb{S}} \int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} |f(z)|^p d\widetilde{v}_\alpha(z) d\sigma(\xi) < \infty \right\},$$

for $p \in (0, \infty)$, $\alpha > -1$.

$$K^{p,q}_{\alpha,\beta} = \left\{ f \in H(\mathbb{B}^m) : \int_{\mathbb{B}} \cdots \int_{\mathbb{B}} \left(\int_{\mathcal{D}(z_1,r)} \cdots \int_{\mathcal{D}(z_m,r)} |f(z)|^p d\widetilde{v}_\alpha(z) \right)^{\frac{q}{p}} d\widetilde{v}_\beta(z) < \infty \right\},$$

for $p, q \in (0,\infty)$, $\alpha > -1$, $\beta > -1$.

$$\begin{aligned} D^p_{\alpha,\beta} \quad = \quad & \Bigg\{ f \in H(\mathbb{B}^m) : \int_0^1 (1-r)^\beta \Bigg(\int_{|z_1|<1} \cdots \int_{|z_{m-1}|<1} \int_{|z_m|<r} |f(z)|^p \times \\ & \times \prod_{j=1}^m (1-|z_j|^{\alpha_j}) dv(z_j) \Bigg) dr < \infty \Bigg\} \end{aligned}$$

for $p \in (0,\infty)$, $\beta > -1$, $\alpha_j > -1$, $j = 1, \dots m$.

These classes were considered in the case of unit disk by many authors (see, for example, [1], [6], [16]).

Functional classes defined with the help of Bergman metric ball and area operator in the unit ball were studied in [3], [10], [11] and also in Chapter 5 and Chapter 6 in [18]. Note that M^p_α coincide with usual weighted Bergman class in ball for $m = 1$ (see [10], [11]).

The following Theorem for $n = 1$ was proved in [7].

Theorem 1. *Let* $m \in \mathbb{N}$, $n \in \mathbb{N}$, $0 < p < \infty$, $\alpha > -(n+1)$ *and* $\gamma = (\alpha + n + 1)m - 1$. *Then* $Trace(M^p_\alpha(\mathbf{B}^m)) = A^p_\gamma(\mathbf{B})$.

Proof. In [4] it was shown

$$\int_0^1 |u(r\xi_1, \dots, r\xi_n)|^p (1-r)^\alpha dr \le C \int_{\Gamma_t(\xi)} |u(z)|^p (1-|z|)^{\alpha-n} dv(z), \tag{3}$$

where $\alpha > -1$, $0 < p < \infty$, $u \in H(\mathbf{B})$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{S}$.

We use this estimate m times by each variable separately and get integrating both sides of obtained estimate by $\mathbf{S}$

$$\begin{aligned} \int_{\mathbf{S}} \int_0^1 \cdots \int_0^1 & |u(r_1\xi_1, \dots, r_1\xi_n, \dots, r_m\xi_1, \dots, r_m\xi_n)|^p \times \\ & \times (1-r_1)^{\alpha_1} \dots (1-r_m)^{\alpha_m} dr_1 \dots dr_m d\xi \\ \le \quad & C \int_{\mathbf{S}} \int_{\Gamma_{t_1}(\xi)} \cdots \int_{\Gamma_{t_m}(\xi)} |u(z_1, \dots, z_m)|^p \times \\ & \times (1-|z_1|)^{\alpha_1 - n} \dots (1-|z_m|)^{\alpha_m - n} dv(z_1) \dots dv(z_m) d\xi, \end{aligned} \tag{4}$$

where $u \in H(\mathbf{B}^m),\ m \in \mathbb{N},\ \alpha_j > -1,\ 0 < p < \infty$.

To prove the estimate we will use the so-called slice functions (see [18], page 125). Let

$$(\widetilde{u}_\xi)(z) = \widetilde{u}(\xi z), z \in \mathbf{D},\ z = |z|\tau, \mathbf{D} = \{w : |w| < 1\},\ \xi \in \mathbf{S},$$

$$u_\xi(z_1, \dots, z_m) = u(\xi_1 z_1, \dots, \xi_n z_1, \dots, \xi_1 z_m, \dots, \xi_n z_m),$$

and

$$z_j = r_j \phi,\ z_j \in \mathbf{D}, j = 1, \dots, m, \widetilde{u}_\xi(z) = u(\xi_1 z, \dots, \xi_n z, \dots, \xi_1 z, \dots, \xi_n z),\ z \in \mathbf{D}.$$

Then $\widetilde{u}_\xi \in H(\mathbf{D})$, and

(5)

$$\int_0^1 \int_0^{2\pi} |\widetilde{u}_\xi(z)|^p (1-|z|)^\gamma d\tau d|z| \le C \sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} \int_0^{2\pi} |\widetilde{u}_\xi(z)|^p (1-|z|)^\gamma d\tau d|z|$$

$$\lesssim C_1 \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left(2^{\frac{-k_1\gamma}{m}} \cdots 2^{\frac{-k_m\gamma}{m}}\right)\left(2^{\frac{-k_1}{m}} \cdots 2^{\frac{-k_m}{m}}\right) \int_{1-2^{-(k_1+1)}}^{1-2^{-(k_1+2)}} \cdots \times$$

$$\times \int_{1-2^{-(k_m+1)}}^{1-2^{-(k_m+2)}} \int_0^{2\pi} |(u_\xi)(z_1, \dots, z_m)|^p (2^{k_1} \cdots 2^{k_m}) d\tau d|z_1| \cdots d|z_m|$$

$$\le C_2 \int_0^1 \cdots \int_0^1 \int_0^{2\pi} |(u_\xi)(z_1, \dots, z_m)|^p \prod_{k=1}^{m} (1-|z_k|)^{\frac{\gamma+1}{m}-1} d\tau d|z_1| \cdots d|z_m|.$$

Using that for $f \in L^1(\mathbf{S}^n, d\sigma)$

$$\int_{\mathbf{S}^n} f d\sigma = \int_{\mathbf{S}^n} d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}\zeta) d\theta, \tag{6}$$

$$f(t\xi) = f(t\xi_1, \dots, t\xi_n, \dots, t\xi_1, \dots, t\xi_n),\ \xi \in \mathbf{S},\ t \in (0, 2\pi), \tag{7}$$

(see [18], Lemma 1.10), we have from (5) integrating both sides of it by sphere $\mathbf{S}$

$$\int_{\mathbf{B}} |\widetilde{u}(z)|^p (1-|z|)^\gamma dv(z) = \sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} \int_{\mathbf{S}} |\widetilde{u}(z)|^p (1-|z|)^\gamma d\sigma(\xi) d|z| \tag{8}$$

$$\le C \int_0^1 \dots \int_0^1 \int_{\mathbf{S}} |u(z)|^p d\sigma(\xi) (1-|z_1|)^{\frac{\gamma+1}{m}-1} \dots (1-|z_m|)^{\frac{\gamma+1}{m}-1} d|z_1| \dots d|z_m|,$$

$\widetilde{u}(z) = u(z, \dots, z)$.

Combining (4) and (8) we have for $0 < p < \infty,\ \gamma > -1$

$$\int_{\mathbf{B}} |u(z,\dots,z)|^p (1-|z|)^\gamma dv(z)$$
$$\le C \int_{\mathbf{S}} \int_{\Gamma_\delta(\xi)} \dots \int_{\Gamma_\delta(\xi)} |u(z_1,\dots,z_m)|^p \times$$
$$\times (1-|z_1|)^{\frac{\gamma+1}{m}-(n+1)} \dots (1-|z_m|)^{\frac{\gamma+1}{m}-(n+1)} dv(z_1)\dots dv(z_m) d\sigma(\xi).$$

If $\beta = \frac{\gamma+1}{m} - (n+1)$, then $\gamma = (\beta+n+1)m-1,\ \beta > -(n+1)$ and so one part of the assertion is proved.

Now we prove the reverse of the last inequality. Let $p \le 1$. We use systematically properties of, so-called, r- lattice $\{a_k\}$ or sampling sequences (see Lemma A and Lemma B, [18]).

For every $s > -1$ we have $F(z,\dots,z) = f(z)$, where

$$F(z_1,\dots,z_m) = C \int_{\mathbf{B}} \frac{f(w)(1-|w|)^s}{\prod_{j=1}^m (1-\langle \overline{w}, z_j\rangle)^{\frac{s+1+n}{m}}} dv(w)$$

by Bergman representation formula (see [18], Theorem 2.11). So we get the following chain estimates ($p \le 1,\ s$ is large enough) using known properties of $\{a_k\}$ from Lemma A and Lemma B(see also [18], Chapter 2) and the fact that $\left(\sum_{k=1}^\infty a_k\right)^p \le \sum_{k=1}^\infty a_k^p,\quad a_k \ge 0,\quad p \le 1.$

$$|F(z_1,\dots,z_m)|^p \lesssim \sum_{k\ge 0} \max_{\mathcal{D}(a_k,r)} |f(w)|^p \left(\int_{\mathcal{D}(a_k,r)} \frac{(1-|w|)^s}{\prod_{j=1}^m |1-\langle \overline{w}, z_j\rangle|^{\frac{s+1+n}{m}}} dv(w) \right)^p$$
$$\lesssim \sum_{k\ge 0} \max_{\mathcal{D}(a_k,r)} |f(w)|^p \frac{(1-|a_k|)^{ps}\,(v(\mathcal{D}(a_k,r)))^p}{\prod_{j=1}^m |1-\langle \overline{a_k}, z_j\rangle|^{\frac{s+1+n}{m}p}}.$$

Then using again the relation $|1-\langle w,z\rangle| \asymp |1-\langle a_k,z\rangle|,\ \ w \in \mathcal{D}(a_k,r),\ \ z \in \mathbf{B}$, (see [18], page 63) and Lemma 2.24 from [18] and Lemma A we finally get

$$|F(z_1,\dots,z_m)|^p \le C \int_{\mathbf{B}} \frac{|f(\widetilde{w})|^p (1-|\widetilde{w}|)^t dv(\widetilde{w})}{\prod_{k=1}^m |1-\langle \overline{z_k}, \widetilde{w}\rangle|^{\frac{s+1+n}{m}p}}, \tag{9}$$

where $t = (n+1+s)p-(n+1),\ t > -1$.

Integrating both sides of the last inequality by sphere we have the following estimate

$$\int_{\mathbf{S}}\int_{\Gamma_{t_1}(\xi)}\cdots\int_{\Gamma_{t_m}(\xi)}|F(z_1,\ldots,z_m)|^p\times$$
$$\times(1-|z_1|)^{\frac{\gamma+1}{m}-(n+1)}\ldots(1-|z_m|)^{\frac{\gamma+1}{m}-(n+1)}dv(z_1)\ldots dv(z_m)d\sigma(\xi)$$
$$\leq C\int_{\mathbf{B}}|f(\widetilde{w})|^p(1-|\widetilde{w}|)^{\gamma}dv(\widetilde{w});\quad \gamma>-1,\quad p\in(0,1].$$

We used m times the following known estimate

$$\int_{\Gamma_t(\xi)}\frac{(1-|z|)^{\nu}}{|1-\langle\overline{w},z\rangle|^{s_1}}dv(z)\leq\frac{C}{|1-\langle\overline{\xi},w\rangle|^{s_1-n-1-\nu}}, \tag{10}$$

where $\xi\in\mathbf{S},\ w\in\mathbf{B},\ \nu>-n-1,\ s_1>\nu+n+1$, (see [13], [14]). It is easy to see that Theorem 1 is proved completely for all $p\in(0,1]$.

For $p>1$, we use the following estimate, which can be easily obtained from Hölder's inequality applied twice and the estimate (see [18], Theorem 1.12):

$$\int_{\mathbf{B}}\frac{(1-|z|)^{\nu}}{|1-\langle w,z\rangle|^{s_1}}dv(z)\leq\frac{C}{(1-|w|)^{s_1-n-1-\nu}},$$

$w\in\mathbf{B},\ \nu>-1,\ s_1>\nu+n+1,$ applied m times for $s_1=\tau_2p'm$ $\left(\tau=p\left(\frac{\nu+n+1}{mp'}-\tau_2\right),\ \ \frac{1}{p}+\frac{1}{p'}=1\right)$

$$|F(z_1,\ldots,z_m)|^p\lesssim\int_{\mathbf{B}}\frac{|f(w)|^p(1-|w|)^s(1-|z_1|^2)^{\tau}\cdots(1-|z_m|^2)^{\tau}}{\prod_{k=1}^m|1-\langle z_k,\overline{w}\rangle|^{pr_1}}dv(w), \tag{11}$$

where $z_j\in\mathbf{B},\ j=1,\ldots,m;\ r_1+r_2=\frac{s+n+1}{m},\ r_1,r_2>0$, and continue as for the case $p\leq1$, using appropriate τ_1,τ_2. The proof of theorem is complete. □

Remark 1. For $m=n=1$ the statement of Theorem 1 is obvious.

Theorem 2. *Let* $n\in\mathbb{N},\ \ 0<p<\infty,\ \ t_j>-1,\ \ \beta_j>-1,$ $j=1,\ldots,m,\ \ \alpha>-1$ *and* $\alpha=\sum_{j=1}^m(\beta_j+2(n+1)+t_j)-(n+1)$, *then* *Trace*$\left(K^{p,p}_{t,\beta}(\mathbf{B}^m)\right)=A^p_{\alpha}(\mathbf{B})$.

Proof. Obviously by Lemma A and B

$$\int_{\mathbf{B}}|\widetilde{f}(z)|^p(1-|z|)^{\alpha}dv(z)\lesssim\sum_{k\geq0}\left(\max_{z\in\mathcal{D}(a_k,r)}|\widetilde{f}(z)|^p\right)(v_{\alpha}(\mathcal{D}(a_k,r))), \tag{12}$$

where $\widetilde{f}(z)=f(z,\ldots,z),\ \ 0<p<\infty,\ \ \alpha>-1.$

We have by Lemma 2.24 from [18] and Lemmas A and B

$$\sum_{k\geq 0} \max_{z\in\mathcal{D}(a_k,r)} |\widetilde{f}(z)|^p C_{k,\alpha+n+1}$$
$$\lesssim \sum_{k_1\geq 0,\dots,k_m\geq 0} \left(\max_{z_j\in\mathcal{D}(a_{k_j},r)} |f(z_1,\dots,z_m)|^p \right) C_{k_1,\dots,k_m,\alpha+n+1},$$

where $C_{k,\alpha+n+1} = v_{\alpha+n+1}(\mathcal{D}(a_k,r))$. Here

$$\alpha = \sum_{j=1}^m \widetilde{\beta}_j + \sum_{j=1}^m t_j - n - 1, \quad t_j > -1,\ j = 1,\dots,m,$$

with $\widetilde{\beta}_j = \beta_j + 2(n+1),\ j = 1,\dots,m.$

$$\|f\|^p_{A^p_\alpha} \leq \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} (1-|z_1|)^{t_1} \cdots (1-|z_m|)^{t_m} \times$$
$$\times \left(\int_{\mathcal{D}(z_1,r)} \cdots \int_{\mathcal{D}(z_m,r)} |f(w_1,\dots,w_m)|^p d\widetilde{v}_{\overrightarrow{\beta}}(\widetilde{w}) \right) d\widetilde{v}(z),$$

where $d\widetilde{v}_{\overrightarrow{\beta}}(\widetilde{w}) = \prod_{k=1}^m (1-|w|^2)^{\beta_k} dv(w)$. At the final step we used Lemma A again and the fact that for $z \in \mathcal{D}(a_k,r)$ $(1-|z|)^t \asymp (1-|a_k|)^t,\ t \in \mathbb{R}$.

Let us prove the reverse to the estimate we obtained above. We again use properties of expanded Bergman projection and we have the following chain of estimates. We have as before for positive large enough integer s,

$$F(z_1,\dots,z_m) = C \int_{\mathbf{B}} \frac{f(w)(1-|w|)^s dv(w)}{\prod_{j=1}^m (1-\langle \overline{w}, z_j\rangle)^{\frac{s+1+n}{m}}};$$

and $F(z,\dots,z) = f(z)$. Hence by Lemma 2.15 from [18], for $p \leq 1$, we have

$$\int_{\mathbf{B}} \cdots \int_{\mathbf{B}} \prod_{k=1}^m (1-|\widetilde{z_k}|)^{t_k} \left(\int_{\mathcal{D}(\widetilde{z}_1,r)} \cdots \int_{\mathcal{D}(\widetilde{z_m},r)} |F(z_1,\dots,z_m)|^p d\widetilde{v}_\beta(z) \right) d\widetilde{v}(\widetilde{z})$$
$$\lesssim \int_{\mathbf{B}} \cdots \int_{\mathcal{D}(\widetilde{z}_1,r)} \cdots \int_{\mathcal{D}(\widetilde{z_m},r)} \int_{\mathbf{B}} \frac{f(w)(1-|w|)^{p(n+1+s)-(n+1))}}{|\prod_{j=1}^m (1-\langle \overline{w}, z_j\rangle)^{\frac{s+1+n}{m}}|^p} \times$$
$$\times \prod_{k=1}^m (1-|\widetilde{z_k}|)^{t_k} \prod_{k=1}^m (1-|z_k|)^{\beta_k} d\widetilde{v}(\widetilde{z}) d\widetilde{v}(\widetilde{w}) d\widetilde{v}(z),$$

where $d\widetilde{v}(\widetilde{z}) = dv(\widetilde{z}_1)\dots dv(\widetilde{z}_m)$. Using the fact that s is large and using the inequality

$$\int_{\mathcal{D}(z_1,r)}\cdots\int_{\mathcal{D}(z_m,r)}\frac{(1-|w_1|)^{\beta_1}\cdots(1-|w_m|)^{\beta_m}}{\prod_{j=1}^m|1-\langle w_j,z\rangle|^{\frac{s+1+n}{m}p}}dv(w_1)\cdots dv(w_m)$$

$$\leq C\prod_{j=1}^m|1-\langle z,z_j\rangle|^{\frac{-(s+1+n)p}{m}+\beta_j+n+1},\quad z\in\mathbf{B},\ z_j\in\mathbf{B},$$

we finally get the estimate we need:

$$\|F\|^p_{K^{p,p}_{t,\beta}}\lesssim\|f\|^p_{A^p_\alpha}, \tag{13}$$

when $\alpha,\ t,\ \beta$ were defined above.

To finish the proof we need the estimate (13) for $p>1$. For that reason we apply estimate (11). Then repeat arguments we provided above for $p\leq 1$ using the fact that s is large enough in Bergman representation formula. The proof of theorem is complete. □

Remark 2. Let us note that for $m=1$ and $n=1$, Theorem 2 is obvious. Our Theorem 2 is new even for $m>1$, $n=1$ (polydisk case).

Theorem 3. *Let* $0<p<\infty$, $\beta>-1$, $\alpha_j>-1$, $j=1,\dots,m$. *Then* $Trace(D^p_{\alpha,\beta}(\mathbf{B}^m))=A^p_{\sum_{j=1}^m\alpha_j+\beta+1+(n+1)(m-1)}(\mathbf{B})$.

Proof. We obtain first a characterization of $D^p_{\alpha,\beta}$ classes via weighted Bergman spaces on polyballs and then we will apply Theorem A and thus we will calculate completely traces of $D^p_{\alpha,\beta}$ classes in polybals.

Let $0<t<1$, $f_t(z)=f(tz)$, $z\in\mathbf{B}\times\cdots\times\mathbf{B}$. Let $r_n=1-2^{-n}$, $n=0,1,\dots$. Using decomposition $\int_0^1P(r)dr=\sum_{k=0}^\infty\int_{r_k}^{r_{k+1}}P(r)dr$, where $P(r)$ is any measurable function on $(0,1)$, it is easy to check

$$\begin{aligned}\|f\|^p_{D^p_{\alpha,\beta}}&=\int_0^1(1-r)^\beta\Bigg(\int_{|z_1|<1}\cdots\int_{|z_{m-1}|<1}\int_{|z_m|<r}|f_t(z_1,\dots,z_m)|^p\times\\&\qquad\times\prod_{j=1}^m(1-|z_j|)^{\alpha_j}dv(z_j)\Bigg)dr\\&\leq C\sum_{n=1}^\infty2^{-n(\beta+1)}\left(\int_{|z_1|<1}\cdots\int_{|z_{m-1}|<1}\int_{|z_m|<r_n}|f_t(z)|^p\prod_{j=1}^m(1-|z_j|)^{\alpha_j}dv(z_j)\right)\\&=C\sum_{n=1}^\infty2^{-n(\beta+1)}A_n^p=K(p,f,\alpha).\end{aligned}$$

We have by Lemma C

$$K \leq C_1 \sum_{n=1}^{\infty} 2^{-n(\beta+1)} \int_{|z_1|<1} \cdots \int_{|z_{m-1}|<1} \int_{r_{n-1}\leq z_m<r_n} |f_t(z)|^p \times$$
$$\times \prod_{j=1}^{m} (1-|z_j|)^{\alpha_j} dv(z_j)$$
$$\leq C_2 \|f_t\|_{A^p(\mathbf{B}^m, dv_{\alpha_1},\ldots,dv_{\alpha_{m-1}}, dv_{\beta+1+\alpha_m})}$$
$$= C_2 \int_{|z_1|<1} \cdots \int_{|z_m|<1} |f_t(z_1,\ldots,z_m)|^p dv_{\alpha_1}(z_1)\cdots dv_{\alpha_{m-1}}(z_{m-1}) dv_{\beta+1+\alpha_m}(z_m).$$

Finally we have

$$\|f_t\|^p_{D^p_{\alpha,\beta}} < C\|f_t\|_{A^p(\mathbf{B}^m, dv_{\alpha_1},\ldots,dv_{\alpha_{m-1}}, dv_{\beta+1+\alpha_m})}. \tag{14}$$

We tend $t \to 1$ and apply Theorem A. So we get

$$Trace(D^p_{\alpha,\beta}(\mathbf{B}^m)) \subset A^p_{\sum_{j=1}^m \alpha_j+\beta+1+(n+1)(m-1)}(\mathbf{B}).$$

The reverse to (14) can be obtained by very similar arguments. So using again Theorem A we get the reverse inclusion

$$A^p_{\sum_{j=1}^m \alpha_j+\beta+1+(n+1)(m-1)}(\mathbf{B}) \subset Trace(D^p_{\alpha,\beta}(\mathbf{B}^m))$$

and the proof is complete. □

3. On traces of Hardy spaces and some functional classes defined via fractional derivatives in polyballs

In this section we will give estimates for traces of classes of holomorphic functions of Besov-type in polyballs defined with the help of fractional derivatives.

For $\alpha \geq 0$, $p \in (0,\infty)$, $\beta \in \mathbb{R}$, $D^\beta f = D^\beta_{z_1}\cdots D^\beta_{z_m} f, r \in (0,1)$, define

$$H^p_{\alpha,\beta}(\mathbf{B}^m) = \{f \in H(\mathbf{B}^m) : \sup_{r<1} \left(\widetilde{M}_p(D^\beta f, r)\right)(1-r)^\alpha < \infty$$
$$\widetilde{M}^p_p(f,r) = \int_{\mathbf{S}} \cdots \int_{\mathbf{S}} |f(r\xi_1,\ldots r\xi_m)|^p d\sigma(\xi_1)\cdots d\sigma(\xi_n).$$

We define Hardy class H^p in polyballs as $H^p(\mathbf{B}^m) = H^p_{0,0}(\mathbf{B}^m)$ for $p \in (0,\infty)$. As usual, we denote by $\overrightarrow{\alpha}$ the vector $(\alpha_1,\ldots,\alpha_n)$. Let

$$A^p_{t,\overrightarrow{\alpha}}(\mathbf{B}^m) = \{f \in H(\mathbf{B}^m) : \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |D^{\alpha_1}\cdots D^{\alpha_m} f|^p (1-|\widetilde{z}|)^t d\widetilde{v}(z) < \infty\},$$

where $\alpha_j \in \mathbb{R},\ j = 1, \dots, m,\quad t > -1,\quad 0 < p < \infty$ and $(1 - |\widetilde{z}|) = \prod_{k=1}^{m}(1 - |z_k|)$. Then the following result can be formulated as a direct corollary of estimates for expanded Bergman projection obtained during the proof of Theorem 1 (see estimate (9)) and some known calculations with fractional derivatives on Bergman kernels that can be found also in [10] and [11].

Theorem 4. (1) *Let* $p \leq 1,\ \alpha \geq \beta \geq 0,$ *then* $A^p_{nm+(\alpha-\beta)pm-(n+1)}(\mathbf{B}) \subset Trace(H^p_{\alpha,\beta}(\mathbf{B}^m))$;
(2) *Let* $p \leq 1,\ \alpha_j \geq 0,\ t \geq \frac{\sum_{j=1}^m \alpha_j}{m},$ *then* $A^p_{mt+m(n+1)-(n+1)-\sum_{j=1}^m \alpha_j}(\mathbf{B}) \subset Trace(A^p_{t,\overrightarrow{\alpha}}(\mathbf{B}^m))$.

Remark 3. Note that for $n = 1$ (polydisk case) and $\alpha = 0,\ \beta = 0$ the first inclusion can be found in [5]. The second inclusion for $\alpha_1, \dots, \alpha_n = 0$ can be also found in [5].

It is not difficult to see as a consequence of Cauchy formula (as in case of polydisk) that the expansion of any function $f,\ f \in H(\mathbf{B}^m)$ can be defined as follows

$$f(z_1, \dots, z_m) = \sum_{n_1 \geq 0} \cdots \sum_{n_m \geq 0} a_{n_1,\dots,n_m} z_1^{n_1} \cdots z_m^{n_m}$$
$$z_j^{n_j} = z_1^{n_j^1} \cdots z_n^{n_j^n},\quad j = 1, \dots, m$$

where $z_k^{n_j^i}$ is in unit disk in $\mathbb{C}$. And hence the corresponding homogeneous expansion of will be defined as follows:

$$f(z_1, \dots, z_m) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} f_{k_1,\dots,k_m}(z_1, \dots, z_m);$$
$$f_{k_1,\dots,k_m}(z_1, \dots, z_m) = \sum_{|s_1|=k_1} \cdots \sum_{|s_m|=k_m} a_{s_1,\dots,s_m} z_1^{\overrightarrow{s_1}} \cdots z_m^{\overrightarrow{s_m}},$$
$$|s_j| = \sum_{i=1}^{n} s_i^j,\quad z_j \in \mathbf{B},$$

and the action of the fractional derivative is given by

$$\left(D^{t_1,\dots,t_m} f\right)(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \prod_{j=1}^{m} (k_j + 1)^{t_j} f_{k_1,\dots,k_m}(z_1, \dots, z_m),$$

with $t_j \in \mathbb{R},\quad j = 1, \dots, m,\quad \left(D^{\overrightarrow{t}} f\right) : H(\mathbf{B}^m) \to H(\mathbf{B}^m)$.

We now provide some estimates for traces of Hardy H^p classes in polyballs. Let $p \geq 2$. Then $\mathrm{Trace}(H^p(\mathbf{B}^m)) \subset A^p_{n(m-1)-1}(\mathbf{B})$, $m \in \mathbb{N}$, $m > 1$, $n \in \mathbb{N}$. The proof follows directly from binomial formula. We will give a short proof for $n = 1$ polydisk case. The general case needs small modification.

Let $\mathbf{D}^n$ be unit polydisk, $dm_{2n}(z)$ Lebesgue measure on $\mathbf{D}^n$. Then we have the following chain of estimates. For $s \in \mathbb{N}$.

$$(k_1 + \ldots + k_n + 1)^s = \sum_{\alpha_j \geq 0; \sum_{j=1}^{n+1} \alpha_j = s} C_s(\alpha_1, \ldots, \alpha_n) \left(\prod_{j=1}^{n} (k_j + 1)^{\alpha_j} \right) \times \\ \times (-1)^{\alpha_{n+1}} (n-1)^{\alpha_{n+1}}.$$

Hence

$$M = \int_{\mathbf{D}^n} |\sum_{k_1,\ldots,k_n} (k_1 + \ldots + k_n + 1)^s a_{k_1,\ldots,k_n} z_1^{k_1} \cdots z_n^{k_n}|^p \times \\ \times (1-|z_1|)^{\alpha_1 p - 1} \cdots (1-|z_n|)^{\alpha_n p - 1} dm_{2n}(z) \\ \leq C \int_{\mathbf{D}^n} |D^{\alpha_1 \cdots \alpha_n} f|^p \prod_{k=1}^{n} (1-|z_k|)^{\alpha_k p - 1} dm_{2n}(z) \leq C \|f\|_{H^p}.$$

The last estimate follows directly from Theorem 4.41 from [18] for H^p, $p \geq 2$ applied n times.

On the other hand obviously by Theorem A we have

$$M \geq C_1 \int_{\mathbf{D}} |\sum_{k_1,\ldots,k_n} (k_1 + \ldots + k_n + 1)^s a_{k_1,\ldots,k_n} z^{k_1 + \ldots + k_n}|^p (1-|z|)^{sp+n-2} dm_2(z) \\ \geq C_2 \int_{\mathbf{D}} \left| \sum_{m \geq 0} (m+1)^s \left(\sum_{k_1 + \ldots + k_n = m} a_{k_1,\ldots,k_n} \right) z^m \right|^p (1-|z|)^{sp+n-2} dm_2(z) \\ \geq C_3 \|\widetilde{f}\|^p_{A^p_{n-2}}, \quad p \geq 2, \quad \widetilde{f} = f(z, \ldots, z).$$

Let us note also that according to well known estimate for Poisson integral of functions from Hardy classes in the unit ball (see [18], page 154) applied $(m-2)$ times we have

$$|f(z, \ldots, z)|^p \leq C \int_{\mathbf{S}^n} \cdots \int_{\mathbf{S}^n} |f(\xi_1, \ldots, \xi_{m-2}, z, z)|^p \times \\ \times \frac{(1-|z|^2)^n \cdots (1-|z|^2)^n}{\prod_{k=1}^{m-2} |1 - \langle \xi_k, z \rangle|^{2n}} d\sigma(\xi_1) \cdots d\sigma(\xi_{m-2}),$$

where $z \in \mathbf{B}$, $f \in H^p(\mathbf{B} \times \cdots \times \mathbf{B}) = H^p(\mathbf{B}^m)$, $0 < p < \infty$.

From last estimate now it is easy to see that if

$$\int_{\mathbf{B}} |f(z,z)|^p (1-|z|)^{n-1} dv(z) \leq C\|f\|_{H^p(\mathbf{B}^2)}, \tag{15}$$

where $0 < p < \infty$, $n \geq 1$, $f \in H^p(\mathbf{B}^2)$. Then for any $m > 1$, $m \in \mathbb{N}$, $f \in H^p(\mathbf{B}^m)$, $0 < p < \infty$,

$$\int_{\mathbf{B}} |f(z,\ldots,z)|^p (1-|z|)^{n(m-1)-1} dv(z) \leq C\|f\|^p_{H^p(\mathbf{B}^m)}. \tag{16}$$

So $Trace(H^p(\mathbf{B}^m)) \subset A^p_{n(m-1)-1}(\mathbf{B})$, $0 < p < \infty$.

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